PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

VOL. XXIII. 1899–1900.

THE 117TH SESSION.

GENERAL STATUTORY MEETING.

Monday, 27th November 1899.

The following Council were elected :—

President.

THE RIGHT HON. LORD KELVIN, G.C.V.O., F.R.S.

Vice-Presidents.

Professor JOHN G. M'KENDRICK, M.D., LL.D., F.R.S.
Professor CHRYSTAL, LL.D.
Sir ARTHUR MITCHELL, K.C.B., LL.D.
Sir WILLIAM TURNER, M.B., F.R.S.
Professor COPELAND, Astronomer-Royal for Scotland.
The Rev. Professor DUNS, D.D.

General Secretary—Professor P. G. TAIT.

Secretaries to Ordinary Meetings.

Professor CRUM BROWN, F.R.S.
Sir JOHN MURRAY, K.C.B., D.Sc., LL.D., F.R.S.

Treasurer—PHILIP R. D. MACLAGAN, Esq., F.F.A.

Curator of Library and Museum—ALEXANDER BUCHAN, Esq., M.A., LL.D., F.R.S.

Ordinary Members of Council.

Sir J. BATTY TUKE, M.D., D.Sc.
A. BEATSON BELL, Esq., Advocate.
Professor SHIELD NICHOLSON, M.A., D.Sc.
Professor JOHN GIBSON, Ph.D.
The Hon. Lord M'LAREN, LL.D.
C. G. KNOTT, Esq., D.Sc.
Dr ALEX. BRUCE, M.A., F.R.C.P.E.
JAMES A. WENLEY, Esq.
The Rev. Professor FLINT, D.D.
JAMES BURGESS, Esq., C.I.E., LL.D.
R. M. FERGUSON, Esq., Ph.D., LL.D.
ROBERT IRVINE, Esq., F.C.S.

Honorary Representative on George Heriot's Trust.

Sir JOHN MURRAY, K.C.B., D.Sc., LL.D., F.R.S.

By a Resolution of the Society (19th January 1880), the following Hon. Vice-Presidents, having filled the office of President, are also Members of the Council :—

HIS GRACE THE DUKE OF ARGYLL, K.G., K.T., LL.D., D.C.L.
SIR DOUGLAS MACLAGAN, M.D., LL.D., F.R.C.P.E.

THE RIGHT HON. LORD KELVIN, President, in the Chair.

Chairman's Opening Address.

(Read December 4, 1899.)

The PRESIDENT, on opening the Session, said—During the past Session 62 papers have been read. Of these, 14 belong to the department of Physics, 10 to Mathematics, 6 to Chemistry, 4 to Oceanography, 1 to Geology, 5 to Natural History, 4 to Comparative Anatomy, 3 to Anatomy, 6 to Physiology, 6 to Meteorology, and 1 to Literature.

Since the commencement of the Session 21 Fellows have been added to our numbers. Of these, 3 are Doctors of Laws or Doctors of Science, 5 are Doctors of Medicine, 4 are Professors.

But during the same period 18 Fellows have been taken from us by death. They include :

Sir JOHN FOWLER, who was a representative of modern railway achievement by his works in England, India, and Egypt, and in conjunction with Sir Benjamin Baker designed the Forth Bridge, the greatest railway bridge which the world has yet seen.

Professor ALLMAN, who held the Chair of Natural History in the University of Edinburgh, whose *magnum opus* is on the Gymnoblastic or Tubularian Hydroids.

Professor RUTHERFORD, who for twenty-five years held the Chair of Physiology in the University of Edinburgh, and whose eminence as a teacher of that science was duly recognised, and led to an extraordinarily large attendance at his lectures.

Sir JOHN STRUTHERS, who was appointed to undertake the duties of the Chair of Anatomy in Edinburgh University in the absence of Professor Goodsir, and who afterwards was Professor of Anatomy in the University of Aberdeen.

Dr JOHN MOIR, who discharged the duties of the Chair of Midwifery in Edinburgh University in the interval which elapsed between the death of Professor Hamilton and the appointment

of Sir James Simpson, and was remarkable for his skill as a Physician.

Mr G. F. Lyster, who was Engineer-in-chief to the Docks of the Mersey, and who designed a system of sluicing for them.

Mr David Chalmers of Redhall, nephew of the great Dr Chalmers, who was deeply interested in this Society, and was also much occupied with antiquarian pursuits.

Mr Robert Cox was Member of Parliament for South Edinburgh, took a great interest in Astronomy, and presented several valuable gifts to the Town Observatory.

Professor Blaikie, who has shown ability as a biographer, and who wrote a small work, entitled *Better Days for the Working Classes*, of which nearly 100,000 copies have been sold.

Mr James Simpson Fleming, who held the responsible position of Cashier and Manager of the Royal Bank of Scotland.

Professor Ewart has entered into an interesting line of research, and given us several remarkable papers on the effects of the crossing of animals, heredity and reversion, which promise concurrently to settle experimentally the vexed question of telegony.

In Physiology, we have had papers on the metabolism due to Fever, by Dr Noël Paton; on the Organs of Ceratodus, by Dr Gregg Wilson; on Changes in the Newt's Stomach during Digestion, by Professor Carlier; on the Life Histories of the Cod and the Whiting, by Dr Masterman; on Duplicitas Anterior, by Dr Bryce; on the Development and Morphology of the Marsupial Shoulder Girdle, by Dr Robert Brown; and on the Restoration of Coordinated Movements after Nerve Section, by Dr Robert Kennedy.

Sir John Murray has given papers on the Temperatures over the Floor and on the Surface of the Ocean, and has favoured us with the results of his Bathymetrical Survey of the Scottish Lakes.

We have had from Dr Flett an exhaustive paper on the Trap Dykes of the Orkneys, in which he confirms the views of Sir Archibald Geikie on the same subject; and from Mr A. C. Seward and Mr A. W. Hill, a paper on the Lepidodendron Stem from the Calciferous Sandstone of Dalmeny.

The Meteorology of Ben Nevis has been further illustrated by Mr Omond, Mr J. Y. Buchanan, and Dr Buchan.

From Prof. A. Crichton Mitchell we have had a paper on the Convection of Heat.

Professor Little has given us a paper on Knots, which used to be a favourite subject with Professor Tait, and treated non-alternate ± Knots of the Tenth Order. We are thankful to Professor Little for a paper of this kind, which involves prolonged labour.

From Dr Muir we have had many papers dealing with abstruse theorems in Determinants.

Sir William Turner has given us papers on the Craniology of certain Tribes of the North-East Frontier of India and of Burma, and on the Decorated Skulls from New Guinea, with their mysterious markings.

Dr Baildon has favoured us with a literary paper—and I wish we had more literary papers—on the Modification of Vowel Sounds by the consonants with which they are in apposition, and has illustrated the subject by the Rimes in the Poems of the Scottish poet Dunbar, of whom it may be said, as of another Scottish poet of the same period:—

> "Still is thy name of high account,
> And still thy verse has charms."

The following Address was presented to Sir George Gabriel Stokes, on the occasion of the Jubilee celebration of his appointment as Lucasian Professor of Mathematics in the University of Cambridge:—

"To Sir GEORGE GABRIEL STOKES, Baronet, Lucasian Professor of Mathematics in the University of Cambridge.

"On behalf of the Council of the Royal Society of Edinburgh, we congratulate you heartily on the approaching completion of the fiftieth year of your tenure of the Lucasian Professorship. We desire to express our conviction that much of the great advance in mathematical and experimental development of Natural Philosophy which has been made in the nineteenth century is directly, or indirectly, due to you. Your published writings on Mathematical and Experimental Physics form an imperishable monument

to your persevering devotion of labour and genius to the increase of knowledge during fifty-seven years.

"We rejoice to know that you enjoy good health and undiminished activity in scientific work. We hope that these may be continued to you for many years to come.

(L.S.) (Signed) "KELVIN, *President.*
(„) "P. G. TAIT, *Secretary.*

"*May* 19*th*, 1892."

Three of the Fellows of the Society—Sir John Murray, Professor D'Arcy Thompson, and Mr Walter E. Archer—were appointed representatives of the British Government at the International North Sea Conference on Northern Fisheries.

We have had, at the request of the Council, three Addresses, of which the first was given by Admiral Makaroff on the construction of a ship, said to be the strongest in the world, made for the Russian Government for the purpose of breaking up the ice which for several months of the year blocks the Russian ports, and he insisted on the desirableness of ascertaining the temperatures and currents of the ocean.

Mr Andrews, of the British Museum, delivered the second special Address, in which he described the Geological Structure of Christmas Island, with its rich deposits of phosphate of lime, and several new genera and species of animals which he found there.

Professor Knott gave the third Address, which was on Earthquake Vibrations, their Propagation through the Earth, and their bearing on the Earth's internal state.

Dr Muir and Lord M'Laren have given Papers developing that branch of Mathematics known as Determinants, and Professor Tait has not been forgetful of Quaternion problems, and has treated of homogeneous strains.

The following brief obituary notices of Fellows of the Society, who have died during last Session, are by no means intended to supersede longer and more complete notices should such be furnished by the relatives and friends of the deceased.

GEORGE JAMES ALLMAN was born at Cork in 1812, and was educated at the Belfast Academical Institution. He took his

degree of M.D. in the University of Dublin, and also in the University of Oxford in 1847. During the year of his graduation he was appointed Regius Professor of Botany in Dublin University, and ten years later he resigned the Dublin chair for that of Regius Professor of Natural History in the University of Edinburgh, with which was incorporated the Keepership of the Natural History Museum. He resigned his Chair in 1870. Allman's first Paper was a botanical one, "On the Mathematic Relations of Cells of Plants." He wrote on the Crinoids, but his greater reputation rests upon his investigations into the Classification and Morphology of the Coelenterata and Polyzoa. His *magnum opus* was on the "Gymnoblastic or Tubularian Hydroids." This monograph ranks among the most perfect and philosophic of all modern zoological treatises. He was one of the most prolific of naturalists, and between the years 1835–1873, and apart from his monographs, produced more than 100 papers. He was elected a Fellow of the Royal Society in 1854, and in 1873 received the Society's Gold Medal. He was elected a Fellow of our Society in 1856, and in 1877 was awarded the Brisbane Gold Medal. In 1878 he was awarded the Cunningham Gold Medal of the Royal Irish Academy, and in 1896 the Gold Medal of the Linnean Society, of which he had been President. He died on 24th November 1898.

Sir James Bain was a native of Glasgow, and was born in the year 1818. He started ironworks at Whitehaven, but always retained his connection with Glasgow. He was elected Lord Provost of Glasgow in 1874. Sir James interested himself much and successfully in extending the dock accommodation of Glasgow. In 1891 he was returned Member of Parliament for Whitehaven. In 1877 he received the honour of knighthood. He took a great interest in scientific matters, and was a Fellow of the Royal Geographical Society and a Fellow of the Scottish Society of Antiquaries. He was elected a Fellow of this Society in 1875, and died on 25th April 1898.

Dr Campbell Black was born in Oban about fifty-five years ago, and loved the Highlands, being at his death President of the Glasgow Gaelic Society, and a member of many other Celtic bodies. He held opinions antagonistic to those of the great majority of medical men, and lost no opportunity of making an

onslaught on what he called Listerism and on Koch's discoveries. One of his favourite sayings was that "Medicine is no more an exact science than millinery." For some years he was Professor of Clinical Medicine in Anderson's College, Glasgow, but owing to his scorn for theories which were held by his colleagues and medical scientists, he was not reappointed in 1897. He was elected a Fellow of this Society in 1896, and died on the 20th December 1898.

Emeritus Professor Blaikie was the son of James Blaikie of Craigiebuckler, Aberdeenshire, advocate, and was born at Aberdeen in 1820. His father was Provost of Aberdeen, and inaugurated the scheme for rebuilding Marischal College. The late Professor was educated at the Aberdeen Grammar School and in Marischal College. He was one of the famous Melvin's most brilliant pupils. In his twenty-third year he was ordained minister of the Parish of Drumblade, but in 1844 he undertook the founding of a new Free Church 'charge' at Pilrig, of which he was the successful pastor for twenty-four years. In 1864 the University of Edinburgh conferred on him the degree of D.D., and in 1872 Aberdeen honoured him with the degree of LL.D. In 1868 he was appointed to the Chair of Apologetics and Pastoral Theology in the New College, Edinburgh, a position which he held for twenty years. He is the author of numerous works on theological and philanthropic subjects, among others of *Heads and Hands in the World of Labour*, and *Better Days for the Working Classes*, of which nearly 100,000 copies were sold. From similarity of name he was frequently mistaken for Professor Blackie, the Professor of Greek, and on one occasion, after a speech by the Greek Professor in praise of the Drama, he received a letter from an Irish female correspondent, saying that as he had recommended his divinity students to attend the theatre she would henceforth leave his publications severely alone. He kept up his scholarship to the end, and after his retirement from his chair spent part of his leisure in translating into Latin verse some of our modern hymns. He was elected a Fellow of this Society in 1862, and died on 11th June 1899.

Mr David Chalmers of Redhall was the son of Mr Charles Chalmers, the founder of Merchiston Castle Academy, and was born at Glasgow in 1820. He was proud of being the nephew of

the great Dr Chalmers. He attended his father's school, and afterwards completed his education at Edinburgh University. He entered into partnership with the Messrs Cowan, papermakers, and subsequently took over the business. He was a Fellow of the Scottish Society of Antiquaries, antiquarian research, indeed, occupying much of his leisure time. He died on 2nd May 1899. He was elected a Fellow of this Society in 1866.

Robert Cox, M.P., was born at Gorgie House in May 1845, and was educated at Loretto School, afterwards at the College Hall, St Andrews, and the University of Edinburgh. In 1892 Mr Cox stood as candidate for the Kirkcaldy Burghs, but was unsuccessful. In 1895 he stood as candidate for South Edinburgh, and gained the seat. He was a man of wide culture, had a considerable knowledge of mechanics, and his love of music induced him to present St Cuthbert's Church with a magnificent organ. He took a deep interest in the development of the City of Edinburgh Observatory, and presented it with a valuable reflecting telescope of 13 inches aperture, equatorially mounted. He married the daughter of Dr Hughes Bennett, Professor of Medicine in the University of Edinburgh. He died on 2nd June 1899. He was elected a Fellow of this Society in 1879.

Dr John Duncan was educated at the High School of Edinburgh, and thereafter graduated with distinction in the University of Edinburgh in 1862. He became a Fellow of the Royal College of Surgeons in 1864, and eventually filled the presidential chair of that body. He was in charge of wards in the Infirmary for twenty years. He gave courses of systematic lectures in the extramural school, and finally attracted one of the largest classes of surgery there. He died on 24th August 1899. He was elected a Fellow of this Society in 1870.

James Simpson Fleming. Born at Forfar in 1828, he began business as a solicitor in Glasgow. In 1854 he accepted the appointment of Law Officer of the Western Bank, and subsequently, when only twenty-nine years of age, he was appointed manager *pro tempore* of the bank, which had to close its doors in 1857. He was one of its four liquidators. From 1853 to 1871 he was a partner in the legal firm in Glasgow of M'Gregor, Stevenson & Fleming, and during nearly the whole of that period he was Secretary of the

Glasgow Chamber of Commerce. About the end of 1871 the Directors of the Royal Bank of Scotland invited him to become their Cashier and General Manager. In 1892 he resigned this office. He died on 8th July 1899. He was elected a Fellow of this Society in 1876.

Sir JOHN FOWLER was the eldest son of the late Mr Fowler of Wadsley Hall, Sheffield. His earliest important appointment was on the Stockton and Hartlepool Railway, of which he was resident engineer. At the age of twenty-seven he was selected as engineer for constructing the large group of railways known as the Manchester, Sheffield and Lincolnshire line, which includes tunnels, viaducts and bridges, in addition to a dock, floating pier, large hydraulic works and steam ferry. Of these vast and multifarious works he had the sole engineering charge. A mere catalogue of the works executed by him from this date would occupy more space than can be afforded here. The Forth Bridge was his greatest work, in the construction of which he was assisted by Sir Benjamin Baker. He must have been gratified in his old age in seeing this and his other works, in full operation, ministering to the social and commercial needs of the country.

In 1866 he was elected President of the Institution of Civil Engineers. In 1885 he was created a K.C.M.G., and in 1890 he was promoted to a baronetcy. In recognition of his services to the science of engineering, the University of Edinburgh conferred on him the degree of LL.D. in 1890. He died on the 20th of November 1898. He was elected a Fellow of this Society in 1887.

Dr JOHN MOIR was born in the French prison of Verdun, for it was there that his father, a naval surgeon, taken prisoner during the Napoleonic wars, was joined by his mother, who remained in captivity with her husband until such time as an exchange of prisoners was effected. He graduated as Doctor of Medicine in Edinburgh in 1828, and became Assistant to Professor Hamilton, predecessor of Sir James Simpson, and conducted the class of midwifery in the University between the death of Hamilton and the appointment of Sir James. He was successively President of the Obstetrical Society, the Medico-Chirurgical Society, and the Royal College of Physicians. He died at the age of ninety-two on 14th May 1899. He was elected a Fellow of this Society in 1865.

Professor WILLIAM RUTHERFORD was born at Ancrum Craig, Roxburghshire, on 20th April 1839. He was educated at Jedburgh Grammar School, and went through the medical course of study in the University of Edinburgh. After a distinguished career as a student, he graduated with honours in 1863, and obtained a gold medal for his thesis. He taught Anatomy for a year in Surgeons' Hall under Dr Struthers. Thereafter he studied at the great Medical Schools of Berlin, Leipzig, Vienna and Paris. In 1865, at the age of twenty-six, he was appointed University Assistant to Professor John Hughes Bennett. In 1869, when only thirty years old, he was appointed Professor of Physiology in King's College, London, and during the last three years of his tenure of that chair he was Fullerian Professor of Physiology in the Royal Institution, London. When Professor Bennett resigned the Chair of Physiology in the University of Edinburgh, Professor Rutherford was appointed his successor. He will probably be judged in the future by his ability as a teacher rather than by devotion to original research, though his work on striped muscle attracted attention both in this country and on the Continent. His knowledge of all branches of physiology was encyclopædic. His principal work was entitled *Actions of Drugs on the Secretion of Bile.* He was also the author of *Outlines of Practical Histology* and a *Text-book of Physiology.* He died on 21st February 1899. He was elected a Fellow of this Society in 1869.

Sir JOHN STRUTHERS was born in 1823 at Brucefield, near Dunfermline. He attended the medical course in the University of Edinburgh, and graduated there in 1845. He was Demonstrator of Anatomy in the University, and was subsequently appointed Lecturer on Anatomy in the Extra-mural School. In 1863 he became Professor of Anatomy in the University of Aberdeen. In that capacity he succeeded in increasing the anatomy accommodation; he had new dissecting-rooms built, he secured a new building for an anatomical museum. He prepared and collected museum specimens, dissections, casts, models, and animal skeletons. In his more advanced course of Osteology he expanded his human into comparative anatomy. In 1889 a failing voice and general weakness induced him to give up his professorship. He then returned to Edinburgh, and took a prominent part in the management of the

hospitals both of Edinburgh and Leith. His contributions to Anatomy are numerous. In 1885 Glasgow University conferred on him the degree of LL.D., and in 1898 the Queen conferred on him a knighthood. He died on 24th February 1899. He was elected a Fellow of this Society in 1894.

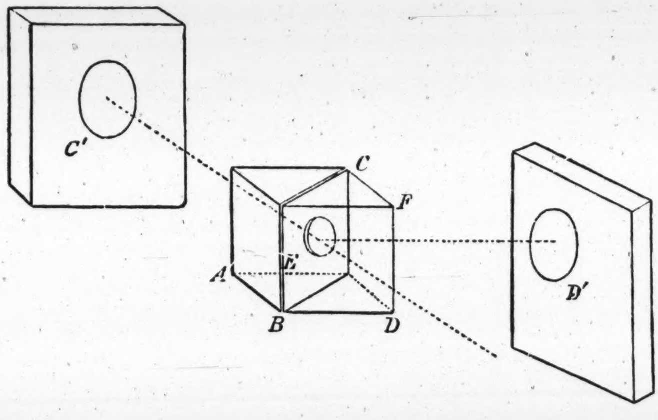
George Williamson, who was elected a Fellow of this Society in 1888, was a member of the Greenock Faculty of Procurators. For over fifty years he performed gratuitously the duties of secretary to the Greenock Infirmary. He was the author of several books dealing with local history, his works entitled *Old Greenock* and *Memorials of James Watt* being his principal productions. He died in his eighty-sixth year.

On Swan's Prism Photometer, commonly called Lummer and Brodhun's Photometer. By Prof. C. G. Knott, D.Sc.

(Read December 19, 1899.)

In 1849 William Swan, subsequently Professor of Natural Philosophy in the University of St Andrews, read a paper on the "Gradual Production of Luminous Impressions on the Eye and other Phenomena of Vision" before the Royal Society of Edinburgh (see *Transactions*, Vol. XVI.). This paper contains some results of high interest, but I have no recollection of ever having seen it referred to in modern literature on the subject.

On April 4, 1859, Professor Swan gave a second paper on the same subject, much briefer than the first, and entirely occupied with descriptions of greatly improved forms of apparatus (see *Transactions*, Vol. XXI.). Among the forms of apparatus de-



scribed is his "Prism Photometer." This is simply and solely the form of photometer described in 1889, exactly thirty years later, by Lummer and Brodhun, and named after them in all recent literature (see *Zeitschrift fur Instrumentenkunde*, Bd. 9). I cannot do better than give Swan's own description in full, and reproduce his own diagram.

He writes:—"An arrangement, which, from an imperfect trial I

have made of it, promises to succeed well for comparing the brightness of the illuminated apertures, may be made by cementing together two equal and similar rectangular glass prisms ABC, BCD, so as to form a parallelopiped, by means of a small portion of Canada Balsam, which, when the prisms are pressed together, expands into a circular thin film E. The illuminated apertures C′, D′, in the screens are placed opposite to the faces AC, CD, and the observer looks through the face BF. The light transmitted through AC, and falling on BC, will be totally reflected, except the portion which falls on the film of Canada Balsam at E, which will be nearly all transmitted to the eye of the observer. The light which is transmitted through the face CD will be totally reflected to the eye by the face BC, except what falls on the Canada Balsam at E, which will be nearly all transmitted. The spot E will appear of a different brightness from the rest of the surface BC, except when the light totally reflected by BC is equal in intensity to the sum of the lights transmitted and reflected at E. The spot E will then disappear, owing to the whole surface of BC, including the spot, becoming uniformly bright. Assuming that the light partially reflected at E has a constant ratio to that totally reflected by the rest of the surface BC, and to that transmitted by AC, it is obvious that the squares of the distances of the flame from the aperture D′ when the spot E disappears will give the ratio of the intensities of the lights transmitted by the aperture C′."

Swan's intention was to publish the results obtained with his improved apparatus; but we can find no record of the continuation of the work. Probably he obtained nothing that materially added to or in any way affected the accuracy of his earlier results; and it was not his habit to write for mere writing's sake.

But whatever may have been the real reason for his subsequent silence, there is not the least doubt that Swan invented, described, constructed, and used, thirty years before the scientific world was ready for it, the prism photometer which Lummer and Brodhun had to re-invent. One of the photometers constructed by Swan himself is now among the apparatus of the Physical Laboratory of Edinburgh University, having been purchased by Professor Tait some years ago along with the best part of Professor Swan's private collection. This photometer is in regular use in the Laboratory.

In the same collection were also two other small prisms intended for the same purpose but not made up. The lid of the small box containing them still bears the inscription in Professor Swan's own handwriting:—"Pair of fine plate-glass prisms made for me by Cooke (1870) for my prism photometer." This inscription, written fully ten years after the first published description, shows that Swan was in the habit of using his photometer.

The fact that Swan had forestalled Lummer and Brodhun in the invention and construction of an ingenious form of photometer has, of course, been familiar to all officially connected with the Edinburgh University Physical Laboratory for some years past. Recently, having occasion to inquire somewhat closely into the history of photometric methods, I determined to make a systematic search through Swan's published papers, which for the most part treat of optical subjects. I had not far to search; for on the plate illustrating the second paper named above I recognised at a glance the prism photometer, and immediately thereafter discovered the descriptive paragraph. My expectation at most was to find some incidental reference to the instrument. To my surprise I found as complete a description of the *essential* instrument as any-one could desire to find. It will remain always a matter of no small astonishment that such an important contribution to knowledge should have escaped the notice of the myriad workers in photometry. In Swan's day there was not the same great interest taken in the subject; but that is no excuse for present neglect.

Swan's photometer was given to a world not ready for its reception. Let us now who know its value not forget that it is "Swan's" photometer.

On the Thermo-electric Properties of Solid and Liquid Mercury. By Dr W. Peddie and A. B. Shand, Esq.

(Read January 8, 1900.)

(*Abstract.*)

By means of a large quantity of solid carbonic acid, obtained from the University Chemical Laboratory, it was found possible to solidify, and maintain in the solid form for a considerable time, a large mass of mercury. Preliminary experiments made about a year ago, in the usual manner, by means of a triple circuit (iron, german silver, mercury), did not give results of a satisfactory kind. This was apparently due to the difficulty of maintaining steady, or steadily varying, temperatures.

Having obtained another supply of carbonic acid about a month ago, the authors made a second attempt. A single iron-mercury circuit was used, and junction temperatures were found by means of iron german-silver thermo-electric circuits. Very much better results were got; and the same arrangement was used to determine, relatively to iron, the thermo-electric position of the mercury when in the liquid state.

The thermo-electric line of the solid metal seems to be very nearly, if not absolutely, continuous with that of the liquid. It intersects the line of 0° C. at a point a little below the intersection of that line by the copper line. It is fairly parallel to the iron line, but intersects it at a point corresponding roughly to the temperature −550° C.

No more definite details are given at present, as the authors intend to repeat the experiment in such a way that the temperatures of the hot and cold junctions can be read simultaneously. In this way they hope to arrive at a very accurate result.

The Torsional Constants of Iron and Steel.

By Dr W. Peddie.

(Read January 22, 1900.)

(*Abstract.*)

This paper gave details of a series of experiments made on the same iron wire as was used in experiments described in previous papers. This series was made on the wire after it had been heated to redness and allowed to cool. A linear relation was again found to hold between log. b and n, where b and n are the quantities (constant in any one experiment) symbolised in the equation $y^n(x+a)=b$, y being range of oscillation and x being number of oscillations of the wire which have taken place since the commencement of the experiment. It was further found that the line representing that relation passed (as did all other such lines previously obtained with this wire) through the point log. $b=2{\cdot}3$, $n=1$. Thus the quantity provisionally called the *Oscillation Constant* in the preceding paper, and regarded as characteristic of the material of the wire, retains its old value even after the wire has been heated to redness.

The present paper contained also a description of a new apparatus now used for the investigation of the phenomena.

It further contained an account of two series of experiments made upon a steel wire. In each series a linear relation held between log. b and n, and the lines representing the relations passed through a point log. $b=3{\cdot}12$, $n=1$. Thus the oscillation constant for steel has a larger value than that for iron.

The theory sketched in last paper was developed a little further, and it was shown how numerical measurements of the elasticity of metals may be obtained from the observations. The deviation from perfectness of elasticity is about six times as great in iron as in steel. The theory shows also that, in all wires of the same material and *pitch* (*i.e.*, ratio of length to radius), the Oscillation Constant has the same value. This indication of theory has not yet been tested experimentally. It is shown also from theory that the Oscillation Constant has an explicit connection with the distortion at which the strongest molecular groups break down.

On the Claim recently made for Gauss to the *Invention* (not the *Discovery*) of Quaternions. By Prof. Tait.

(Read December 18, 1899.)

It is only within a few months that my attention has been (at first accidentally) called to this matter. For, though I owe to the kindness of Prof. Klein a copy of his and Sommerfeld's *Theorie des Kreisels*, I had passed over, in reading the work, the "Digression on Quaternions" which it contains. But Prof. C. N. Little, in the course of correspondence about his remarkable paper on *Knots* (whose passage through the press I was looking after), referred me for a numerical detail to an article by Prof. Klein on the progress of publication of Gauss' *Gesammelte Werke*. Shortly afterwards Prof. Joly called my attention to the same article from another point of view. These references have led me to write the present paper; whose somewhat puzzling title is explained in the first section below.

1.

In 1894 a paper by Prof. Cayley was read before the Society, under the title "*Coördinates* versus *Quaternions*." In this paper the gain in compactness and expressiveness secured by the use of the quaternion method was allowed; but the concession was virtually nullified by the implication that, to be of any use, these simple expressions must be degraded into the vile elements of x, y, z or i, j, k, which were looked upon as their *necessary* basis.

In reply, I allowed that this statement was to a certain extent warranted, provided the quaternion were regarded as Hamilton's brilliant *Invention* of 1843:—a splendid system of imaginaries; but insisted that it had no application whatever to the quaternion of the latter half of the century:—a *Discovery* of the highest order, in which the *Real* took everywhere the place of the *Imaginary*. From that point of view, of course, the discovery was the great thing, the invention merely an exceedingly elegant trifle. Still *both* were regarded as due exclusively to Hamilton.

These two papers were printed in our *Proceedings*, vol. xx.

2.

But Prof. Klein, in the last published part of Klein u. Sommerfeld, *Ueber die Theorie des Kreisels*, p. 512, has repeated a statement made by him in the *Mathematische Annalen* (li. 128) to the effect that Gauss must, in future, be looked upon as, at least in some sense, the *Inventor* of quaternions. Here are the passages, the only hints as to the contents of this portion of Gauss' *Nachlass* which it seems are to be given until the publication of his *Gesammelte Werke*, Bd. VIII. I translate freely.

" . . . and further, that the bases (*Grundlagen*) of the Quaternion-theory are explicitly contained in the incidental notes (*gelegentlichen Aufzeichnungen*) of Gauss. In support of this surprising result we quote a few statements from a preliminary communication about the publication of Gauss' *Works* . . . . ."

" . . . And, what may appear even more startling, he had in 1819 exhibited what he calls the *Mutationen des Raumes* (Turnings of Space round the origin of coördinates, coupled with general Dilatation), by means of the same four parameters which are employed in the subsequent quaternion-theory; he calls the group of them *Mutationsskala*, and gives explicitly the formulæ for the composition of two *Skalen* (that is, the multiplication of two quaternions), using the symbolic form of writing

$$(a\ b\ c\ d) . (\alpha\ \beta\ \gamma\ \delta) = (A\ B\ C\ D);$$

and expressly remarks that we are dealing with a non-commutative process!"

[Obviously, if these refer to quaternions at all, it is to their *original*, i.e., *invented*, form alone.]

The note of exclamation is due to Prof. Klein. Its presence is puzzling, for certainly no one can imagine that a Gauss was required to discover that rotations are not, in general, commutative; nor even that a *Drehstreckung* (the above combination of rotation and dilatation) depends upon four numbers.

In the *first* part of this work of Klein and Sommerfeld there is a *Digression on Quaternions*, in which the *Drehstreckung* is directly identified with a quaternion. In fact, at p. 58 we find the following statements:—

"*Eine Quaternion bedeutet nichts anderes als die Operation der Drehstreckung.*"

"*Eine gewöhnliche Drehung ist eine Einheitsquaternion.*"

Hence, of course, the claim made for Gauss to at least a share in the invention of quaternions.

Unfortunately for such a conclusion, a *Drehstreckung* is *not* a Hamiltonian quaternion at all, but a totally different kind of concept. It is obviously only a very limited form of linear and vector operator (kinematically a *strain*) depending upon *four* constants instead of the usual *nine*; and might, perhaps (but on *that* account solely), have been designated by the name quaternion, had the name not been already more worthily bestowed.

3.

A quaternion, as Hamilton gave it, forms an indispensable part of any conceivable complete theory of vectors. It expresses the relation of one vector to another, or supplies the factor required to convert one into the other. It is completely determined *by these two alone*, and is thus a conception as real as either. In this sense it was called by Hamilton a *Biradial.* It has a plane (or rather an *aspect*), an angle, and the ratio of the lengths of its two legs; and all biradials characterized by like conditions of these kinds are regarded as equivalent to one another. [Equality of angles implies that they are to be measured in the same *sense*.] A quaternion, therefore, when applied to any vector *in or parallel to its own plane*, turns it through a given angle in or parallel to that plane, and alters its length in a given ratio. When the legs of the biradial are equal, and its angle a right angle, the quaternion (as Hamilton showed) is fully represented by the unit-vector perpendicular to its plane. All these particular statements are contained in the general expression

$$q = \beta/\alpha = \frac{b}{a}(\cos A + \epsilon \sin A),$$

where β and α are the vector legs of the biradial, b and a their lengths, A its angle, and ϵ the unit-vector perpendicular to its plane. Obviously, when this is applied to a vector which is not perpendicular to ϵ, the result is a new *Quaternion, not* a vector.

4.

In its initial conception the quaternion had no direct connection whatever with rotation. But, of course, as an organ of expression capable of dealing with all space-problems, it can be employed to describe the effect of rotation.

Thus, if we are to represent the effect of turning a vector ρ (conically) round an axis ϵ (a unit-vector) through an angle A, it is obvious that ρ must be resolved into components parallel and perpendicular to ϵ. Of these the first is unaltered, the second is made to rotate round ϵ through the angle A. Hence, if ϕ be the *operator* (*not*, it is to be carefully observed, a *multiplier*) which produces the rotation, we have, since

$$\begin{aligned}\rho &= -\epsilon S\epsilon\rho - \epsilon V\epsilon\rho,\\ \phi\rho &= -\epsilon S\epsilon\rho - (\cos A + \epsilon\sin A)\epsilon V\epsilon\rho\\ &= \rho\cos A - \epsilon S\epsilon\rho(1-\cos A) + V\epsilon\rho\sin A.\end{aligned}$$

If we multiply this by e (the conjoined dilatation) the right hand side represents the effect of a *Drehstreckung* on any vector ρ. I say *effect*, because a *Drehstreckung* is not a space-reality like a quaternion, it requires a subject before it can obtain embodiment.

Introducing, instead of A, a scalar w, such that

$$\sin A = \frac{2w}{w^2+1}, \quad \cos A = \frac{w^2-1}{w^2+1}; \text{ or } w = \cot\tfrac{1}{2}A;$$

and remembering that

$$\epsilon^2 = -1$$

in this case, we have

$$\begin{aligned}\phi\rho &= \frac{1}{w^2-\epsilon^2}\Big((w^2+\epsilon^2)\rho - 2\epsilon S\epsilon\rho + 2wV\epsilon\rho\Big)\\ &= \frac{1}{w^2-\epsilon^2}\Big\{(w+\epsilon)\rho(w-\epsilon)\Big\}\end{aligned}$$

If we write r for the quaternion $w+\epsilon$, this becomes

$$\phi\rho = r\rho r^{-1}$$

a remarkably simple expression given by Hamilton (*Proc. R.I.A.*, Nov. 1844), and shortly afterwards by Cayley (*Phil. Mag.*, Feb. 1845). This shows that Gauss's *Drehstreckung*, like everything else in space, *can be represented by means of quaternions*, but in *its*

case as a quaternion operator, not as a quaternion. And it is specially to be noted that the angle of the quaternion r is only the *half* of that of the *Drehstreckung.*

5.

The utter difference in kind between the two concepts comes out even more clearly when we consider the vector data necessary to specify them respectively.

To determine, fully, a *Quaternion,* requires but TWO vectors. This would ordinarily involve six scalar conditions; but two of these are not required, because the aspect and angle and the ratio of the legs of the biradial are the sole essentials :—the orientation of the biradial in its own plane, and its scale of size, being immaterial.

To determine a *Rotation* we must have TWO PAIRS of vectors, but there are other specifications, or necessary limitations, as to their lengths, etc., which reduce the number of really necessary and independent scalar data to three. These will be obvious from the results of the subjoined analysis. [What is essentially requisite amounts to two pairs of points on the unit sphere, those of each pair having the same arcual distance. This is at once apparent when we consider the nature of the possible displacements of a cap which fits a sphere, and which has, therefore, three degrees of freedom only. Of course the factor for Dilatation makes up the Tetrad required for the *Drehstreckung.*]

Let $\phi\alpha = \beta,\ \phi\alpha_1 = \beta_1$, or as above

$$r\alpha = \beta r,\ r\alpha_1 = \beta_1 r,$$

so that we must have $T\alpha = T\beta,\ T\alpha_1 = T\beta_1$.

[Hence, by the way, $r\alpha\alpha_1 = \beta r . \alpha_1 = \beta . r\alpha_1 = \beta\beta_1 r$; which shows that the data are at least *sufficient*; and that $S\alpha\alpha_1 = S\beta\beta_1$.]

We have $S(\beta - \alpha)r = 0,\ S(\beta_1 - \alpha_1)r = 0$, so that

$$Vr = xV(\beta - \alpha)(\beta_1 - \alpha_1).$$

But

$$\beta(Sr + Vr) = (Sr + Vr)\alpha.$$

Substitute the above value of Vr, and we have

$$\begin{aligned}(\beta - \alpha)Sr &= x(V(\beta - \alpha)(\beta_1 - \alpha_1) . \alpha - \beta V(\beta - \alpha)(\beta_1 - \alpha_1)) \\ &= x(\beta - \alpha)(S(\beta_1 - \alpha_1)\alpha + S\beta(\beta_1 - \alpha_1)) \\ &= x(\beta - \alpha)S(\alpha + \beta)(\beta_1 - \alpha_1)\end{aligned}$$

Thus, finally,

$$r = x\Big(\mathrm{S}(\alpha+\beta)(\beta_1-\alpha_1)+\mathrm{V}(\beta-\alpha)(\beta_1-\alpha_1)\Big)$$

where x is, of course, indeterminate. This value may be put in a great variety of other forms, in consequence of the necessary relations amongst α, β, α_1 and β_1; all of which may obviously be regarded as unit-vectors. Perhaps the simplest of these is

$$r = x(\beta(\beta_1-\alpha_1)+(\beta_1-\alpha_1)\alpha).$$

6.

Thus, generally, the expression for a *Drehstreckung* in terms of the *necessary* data is

$$e(\beta(\beta_1-\alpha_1)+(\beta_1-\alpha_1)\alpha)(\quad)\frac{1}{\beta(\beta_1-\alpha_1)+(\beta_1-\alpha_1)\alpha}.$$

This is in all respects in marked contrast to the extremely simple expression for a *Quaternion* in terms of *its* necessary data, viz., as above,

$$\beta/\alpha.$$

Treating for a moment β and α as unit vectors (for we may at once do so by neglecting the tensors, which are mere numbers, commutative with everything), a *unit Quaternion* presents itself as

$$\beta/\alpha \quad \text{or} \quad -\beta\alpha,$$

and a *Rotation* as

$$+\beta\alpha(\quad)\alpha\beta.$$

Their respective effects are:—

on α, $\qquad \beta, \qquad$ and $-\beta\alpha\beta = -\alpha - 2\beta\mathrm{S}\alpha\beta$;
on β, $\qquad -\beta\alpha\beta, \qquad$ and $\beta\alpha\beta\alpha\beta = +\beta(4\mathrm{S}^2\alpha\beta-1)+2\alpha\mathrm{S}\alpha\beta$;
and on $\qquad \mathrm{V}\alpha\beta = \frac{1}{2}(\alpha\beta-\beta\alpha)$,
they are $\qquad \beta\alpha\mathrm{S}\alpha\beta - 1, \qquad$ and $\quad +\mathrm{V}\alpha\beta$.

In the case of the rotation the results are, of course, all vectors; but the quaternion necessarily changes $\mathrm{V}\alpha\beta$ into a quaternion, because that vector is perpendicular to its plane.

7.

With regard to Prof. Klein's statement that Gauss had explicitly given the formula for the multiplication of two quaternions, it is

sufficient to state that since we NOW know that a *Drehstreckung* is symbolically expressed in quaternions by

$$er(\quad)r^{-1},$$

the resultant of two successive operations of this kind is necessarily

$$ee_1qr(\quad)r^{-1}q^{-1},$$

or

$$ee_1(qr)(\quad)(qr)^{-1};$$

i.e., it INVOLVES qr in the same extremely novel and peculiar manner as do the separate operators *involve* q and r respectively. Thus the multiplication of quaternions can be identified with the superposition of two *Drehstreckungen* in the same (erroneous) sense only as that in which a quaternion itself is identified with a *Drehstreckung*.

It is most specially to be observed that Prof. Klein does *not* claim for Gauss any knowledge of how to *add* quaternions, simple and direct as the process is. How could Gauss have missed such an obvious matter if his *Drehstreckung* had been really a quaternion? In fact, the sum of two *Drehstreckungen* is not, in general, a *Drehstreckung*; though it is, of course, a linear and vector operator. To add two *Drehstreckungen* they must first be embodied, separately, in any common vector, and the resulting vectors *geometrically* compounded. Then the *Drehstreckung* (if there be such) which produces the resultant from the original vector must be found. Take a very simple case. Obviously we have

$$-ei\rho i - e_1 j\rho j = (e_1 - e)(iSi\rho - jSj\rho) + (e_1 + e)kSk\rho.$$

The terms on the left are *Drehstreckungen*, applied to a common vector ρ. The right is *not* an embodied *Drehstreckung* but a linear and vector function of ρ, which, in the particular case of $e_1 = e$, *reduces space to an infinite straight line!*

To add two *Quaternions* is a mere *algebraical* operation, for they do not require embodiment.

Euler and Gauss, of course, easily anticipated Rodrigues in the mere expression of the conical rotation from one set of rectangular axes to another. But between that and the recognition of the quaternion (even as *invented* only) "there is a great gulf fixed"; and the passage across it was due *entirely* to Hamilton.

Professor Klein's View of Quaternions; a Criticism. By Prof. C. G. Knott.

(Read December 18, 1899.)

In the first part of Klein and Sommerfeld's treatise "Ueber die Theorie des Kreisels," there is a section entitled, *Excurs über die Quaterionentheorie.* In the preceding paper, Professor Tait has discussed the main conclusion contained in this digression; and I here propose to sketch the line of argument by which Klein and Sommerfeld have arrived at their curious mis-interpretation of Hamilton's Quaternion.

In Chapter I. (*Die Kinematik des Kreisels*) the authors discuss the analytical representation of the rotations involved in the motions of a top of which one point is fixed. On page 21, they introduce four parameters A, B, C, D, satisfying the condition that the sum of their squares is unity. These are defined in terms of four other quantities, which have already been defined in terms of the well-known asymmetric representation by means of Euler's angles θ, ϕ, ψ. In terms of these angles, A, B, C, D have consequently the values

$$A = \sin\frac{\theta}{2}\cos\frac{\phi-\psi}{2} \qquad C = \cos\frac{\theta}{2}\sin\frac{\phi+\psi}{2}$$

$$B = \sin\frac{\theta}{2}\sin\frac{\phi-\psi}{2} \qquad D = \cos\frac{\theta}{2}\cos\frac{\phi+\psi}{2}$$

They have also (p. 38) the values

$$A = \sin\frac{\omega}{2}\cos a \qquad C = \sin\frac{\omega}{2}\cos c$$

$$B = \sin\frac{\omega}{2}\cos b \qquad D = \cos\frac{\omega}{2}$$

where $\cos a$, $\cos b$, $\cos c$, are the direction cosines of the axis of rotation, about which the single rotation through angle ω is the rotation determined by the angles θ, ϕ, ψ.

Hence the quantities A, B, C, D correspond to Cayley's B, C, D, A in his *Philosophical Magazine* paper of 1845, and are

identical with the quantities x, y, z, w used by Tait in his expression (Tait's *Quaternions*, § 375) for the quaternion

$$q = xi + yj + zk + w$$

in terms of which the rotation is symbolised by Hamilton's remarkable form

$$q(\quad)q^{-1}$$

Klein and Sommerfeld call the quantities A, B, C, D the *Quaternionengrössen* (p. 21), and speak of them as supplying the transition to Hamilton's Theory of Quaternions. This seems to be, at first reading, correct enough; for undoubtedly the quantity

$$Ai + Bj + Ck + D$$

is a Hamiltonian Quaternion *when i, j, k are used in the Hamiltonian sense.*

But now let us pass to § 7, pp. 55–68, and consider carefully the authors' *Excurs über die Quaternionentheorie.*

In the first place the "Drehstreckung" is introduced, being defined as "an operation which is compounded of a rotation about the origin O, and an isotropic expansion with reference to O." If the length of every line is changed in the ratio $T:1$, then the Drehstreckung can be symbolised by the four magnitudes A, B, C, D, which, however, instead of having the sum of their squares equal to unity, satisfy the equation

$$A^2 + B^2 + C^2 + D^2 = T$$

Two Drehstreckungen acting in succession produce a resultant Drehstreckung, and the equations connecting the twelve quantities of the type A, B, C, D, are obviously the same as those that hold when the Drehstreckungen are simple rotations ($T = 1$). These are given, and then the authors say: "The primitive (*ursprüngliche*) definition of the word quaternion we base on our conception of the Drehstreckung: *A quaternion signifies nothing else than the operation of the Drehstreckung.* It is completely determined by the magnitude of the Streckung (T), by the axis of the rotation (a, b, c) and the magnitude of the half-angle of rotation $\left(\frac{\omega}{2}\right)$."

The Drehstreckung Q, determined by the four magnitudes A, B, C, D, is then written in the form

$$Q = iA + jB + kC + D$$

in which i, j, k are carefully described as three imaginary units; their introduction being "etwas rein conventionelles." Further, "The magnitude T is called, after Hamilton, the *Tensor of the Quaternion.* Therefore we may say: *An ordinary rotation is a unit quaternion* (i.e., *a quaternion of tensor unity*)."

Already Klein and Sommerfeld have parted company with Hamilton; for, although, with Hamilton's meanings of i, j, k, Q is a quaternion, the tensor of the *quaternion* Q is not T, but is $\sqrt{T}$, and a quaternion can never be an "ordinary rotation."

The geometrical meaning of the quantity $Ai+Bj+Ck+D$ we know, provided i, j, k are used in the Hamiltonian sense; and, as will be seen later, Klein and Sommerfeld, in spite of guarded statements about their purely conventional character, do really use them in Hamilton's sense whenever there is any analytical work to be done. Then, again, the operation called the Drehstreckung we also know, for it is a simple modification of an ordinary rotation. But to assert the identity of quaternion and rotation, and to symbolise the latter by means of an expression appropriate to the former,—that surely is a misuse of the mathematical term identity, and a playing fast and loose with the recognised principles of mathematical symbolism.

It is important from the outset to recognise this duality or ambiguity of significance attached to the symbol Q. For some purposes it is treated as a quaternion, and for others as a Drehstreckung. The avowed aim of the authors is to show that Hamilton's quaternion is nothing else than a Drehstreckung, the name given by them to a conception which, as we learn from the last page of Part ii. of their Treatise, was first distinctly described by Gauss. Yet no one who really knows what a quaternion is could for a moment admit the identity. To find anything at all comparable to this attempt to identify two fundamentally different conceptions, we should have to go to old literatures in which the uncritical editor has pieced together into a kind of historic mosaic two traditions from quite different sources. As a foundation on which to build a mathematical superstructure, Klein and Sommerfeld's *Excurs über die Quaterniontheorie* suggests the iron and clay feet of Nebuchadnezzar's image. Happily they do not try to advance their mathematical idol beyond the visionary

stage; for, as they admit on p. 66, they "have no occasion in succeeding chapters to return to quaternion calculation."

Meanwhile, having asserted the identity of quaternion and rotation, the authors proceed to adopt Hamilton's nomenclature, calling D the scalar part and $(iA+jB+kC)$ the vector part of the quaternion (Drehstreckung?).

They then consider a quaternion which is reduced to its vector part, and which is identified with a Drehstreckung whose angle of rotation is $\omega=\pi$, that is, two right angles. This special kind of Drehstreckung, this semi-revolution about an axis, combined with isotropic expansion, is called a *Wendestreckung.* Regarded as a Wendestreckung the vector is assumed to take the analytical form

$$V=iX+jY+kZ$$

But if this be a Wendestreckung, so also is the quantity $iA+jB+kC$, which, on their assumptions, is an important part of the Drehstreckung Q. This no doubt is the Wendestreckung to which the Drehstreckung Q is reduced when $\omega=\pi$. But, when associated with the so-called scalar in the complete expression for the *Drehstreckung*, the so-called vector cannot be interpreted in any sense as a *Wendestreckung.* The most elementary considerations in the geometry of rotations show that, in its effect upon a body, the assumed analytical expression for the Drehstreckung must be treated as a whole. The expression, in fact, is *non-distributive.* Thus $v(iA+jB+kC+D)$, where v is a vector line and the part in brackets a *Drehstreckung*, cannot be expanded in the form $viA+vjB+vkC+vD$. Nevertheless the authors assert (p. 59) that two quantities of the form Q may be added together as Hamiltonian quaternions are added—*i.e.*, the distributive law, which holds for true quaternions, is assumed to hold also for Drehstreckungen. But this assumption is inadmissible; for, as a matter of fact (see Professor Tait's foregoing paper, p. 23), two Drehstreckungen when *added* together cannot in general be represented as a single Drehstreckung.

Throughout pp. 59–62 the quantities of the form Q and V are treated analytically exactly as Hamilton's quaternions and vectors are treated. Thus, in order that the magnitudes A'' B'' C'' D'' which constitute Q'' $(=QQ')$ may be properly related to the

corresponding magnitudes that constitute Q and Q', the "three imaginary units" i, j, k must *of necessity* fulfil Hamilton's equations—

$$i^2 = j^2 = k^2 = -1$$
$$ij = k, \; jk = i, \; ki = j$$
$$ji = -k, \; kj = -i, \; ik = -j.$$

In like manner the product of two vectors

$$vv' = (ix + jy + kz)(ix' + jy' + kz')$$

leads to Hamilton's well-known scalar and vector products; and the usual geometrical meanings of these are given with reference to the vectors which enter into them.

Thus, according to Klein and Sommerfeld, $i. j, k$ are vectors as well as imaginary units; and they are also regarded as Wendestreckungen of tensor unity (p. 61), that is, as operators producing a semi-revolution (*Umklappung*) round an axis. They say:—"The resultant of two semi-revolutions about the same axis is identity; two semi-revolutions about mutually perpendicular axes give a semi-revolution about the normal to the two axes. If we wish to make the algebraic sign right, we must, as on p. 36 and following, pass from the consideration of the whole to that of the half angle of rotation. Then we recognise: it will be $i^2 = -1$, because i^2 has to do with a whole revolution, whose half angle of rotation to modulus 2π is equal to π (and not equal to zero). Moreover, the formulæ (8) $[i^2 = j^2 = k^2 = -1]$ recall the equation $i^2 = -1$ in the theory of ordinary complex numbers."

The reference to p. 36 is simply a reminder that the expressions A, B, C, D, involve $\sin\frac{\omega}{2}$ and $\cos\frac{\omega}{2}$ and not $\sin\omega$ and $\cos\omega$, and that there are difficulties in regard to the signs.

But if, in any true symbolic sense, i is to represent a semi-revolution about an axis, and if, following Klein and Sommerfeld's notation, we represent the semi-revolution of the body B about the x-axis by the symbol Bi, have we not good reason to expect that Bii should be equal to B, *i.e.*, $i^2 = +1$? Klein and Sommerfeld say distinctly that Bi^2 is identical with B; and yet i^2 is also to be equal to -1, because of half angle considerations and the theory of complex numbers! This "facing both ways" of i^2

springs from the attempt to make a quaternion mean a rotation. A mathematical Janus has come into being. Had the authors realised or distinctly stated that their i, j, k are not always associative, so that $Bi.i$ is not the same as $B.ii$, they might have saved their readers considerable confusion; but then their i, j, k would no longer have been the same as Hamilton's, and they could not, with any show of propriety, have used the term Quaternion at all.

Cayley showed in 1845 (see *Phil. Mag.*) that the four scalar quantities in the quaternion $iA+jB+kC+D$ were the quantities symmetrically involved in Rodrigues' expressions defining the rotation

$$(iA+jB+kC+D)(\quad)(iA+jB+kC+D)^{-1},$$

and some further investigations are given in a later paper (*Phil. Mag.*, 1848). The question is also treated in Tait's paper "On the Rotation of a Rigid Body" (*Trans. Roy. Soc. Edin.*, 1868; *Scientific Papers*, vol. i. p. 99). Klein and Sommerfeld's innovation is to make $iA+jB+kC+D$ symbolise the rotation, or, more generally, the Drehstreckung.

Passing on now to the *analytical* part of their discussion, we are introduced to the vectors

$$v=ix+jy+kz \text{ and } V=iX+jY+kZ$$

which are such that the turning part of the Drehstreckung Q changes the direction of v into the direction of V, while its tensor part (T) changes the length of V into the length of v; in symbols

$$vQ=VT^2,$$

where for simplicity V is understood to have unit length. Here v and V are simply directed lines.

The next step, however, is to consider v and V as *Wendestreckungen* and to combine them in a particular way with the Drehstreckung Q. The result of the investigation, which extends over nearly two pages, is the demonstration of the formula

$$(\quad)vT^{-1}=(\quad)QVQ^{-1},$$

where the empty bracket represents any system acted upon by the operators v and Q.

But this equation is simply equivalent to a quaternion identity.

For, writing $q(\quad)\mathrm{K}q$ as one Hamiltonian form and unambiguous symbolic equivalent of Klein and Sommerfeld's Drehstreckung Q, $(\mathrm{T}q)^2$ being equal to their T, we find, putting α and β instead of the *vectors* v and V, that the equation $vQ = VT^2$ takes the form

$$q\alpha\mathrm{K}q = \beta(\mathrm{T}q)^4.$$

Also the symbol $(\quad)QVQ^{-1}$ is, in quaternion symbolism,

$$q^{-1}\beta q(\quad)\mathrm{K}q\mathrm{K}\beta\mathrm{K}q^{-1} = q^{-1}q\alpha\mathrm{K}qq(\quad)\mathrm{K}qq\mathrm{K}\alpha\mathrm{K}q\mathrm{K}q^{-1}(\mathrm{T}q)^{-8}$$

$$= \alpha(\quad)\mathrm{K}\alpha(\mathrm{T}q)^{-4},$$

the required result.

It is well to note here that, although v and α are the same *vectors*, $\alpha(\quad)\mathrm{K}\alpha$ and Klein and Sommerfeld's Wendestreckung v are not quite the same *operators*. Their tensors differ, the Wendestreckung v being equivalent to $(\mathrm{T}q)^{-2}\alpha(\quad)\mathrm{K}\alpha$.

Immediately following the demonstration of the equation just discussed there is given on pp. 64–65 an analytical investigation essentially the same as that given long ago by Cayley, from which the direction cosines of the new positions of a set of rectangular axes with reference to the original positions are expressed in terms of the quantities A, B, C, D. This investigation is of course quite correct, because, *for the moment*, the authors use the quantities Q, v, V really in their true quaternion significations and not as Drehstreckungen.

Thus, in the analytical part of their work, Klein and Sommerfeld simply reproduce long known results and follow accurately Hamilton and Tait. But they leave true quaternion lines when they regard Q $(=iA+jB+kC+D)$ as a complete symbol for the operation which they call a Drehstreckung. In the symbolic equation

$$(\quad)\,vT^{-1} = (\quad)QVQ^{-1}$$

Q, V, v are rotations or very particular types of strain. They are neither true quaternions, nor true vectors. Yet, for reasons which are plain to the quaternionist, these Drehstreckungen depend in a most intimate manner upon the quaternion $(iA+jB+kC+D)$ and the vectors $(ix+jy+kz)$ and $(iX+jY+kZ)$. In all this there is nothing new. Nevertheless the authors proceed to claim that their "geometrical definition of the Drehstreckung leads to a

complete, clear and comprehensive conception of the quaternion calculus. It has, in addition, the advantage of indicating clearly the sphere of applicability (*Anwendungsgebiet*) of quaternions. . . . Quaternions will be in place when we wish to have a convenient algorithm for the combination of rotations and dilatations." If that were all, the quaternion might as well have never existed; for a Drehstreckung is not a very practical dynamic conception, although the rotation is of fundamental importance. It has, of course, been long recognised by workers in quaternions that the quaternion method lends itself powerfully to the treatment of *all* kinds of strains; but because it is peculiarly fitted to attack general problems in the rotation of a rigid body, it does not necessarily follow, as Klein and Sommerfeld seem to suggest, that its value in other directions is insignificant.

Regarding Hamilton's definition of a quaternion as the quotient of two vectors, Klein and Sommerfeld remark:—"As the basis of a theory this definition is scarcely adapted to the end aimed at; for the expression 'quotient of two vectors' requires first an explanation of itself, and, unless that be given,* diverts our attention wholly to a vague (*unklar*) analogy with the rules of ordinary algebra. The definition may, of course, be theoretically justified, and has indeed certain advantages, to be mentioned immediately; but it does not seem appropriate to begin with it."

To this expression of an opinion—and it is little else—the natural reply is, Why not? Is Hamilton's "Quotient of two Vectors" the only expression in mathematics that requires to be explained? Hamilton, indeed, carefully guarded his readers against reading into the meaning of the word "quotient" more than is essentially involved in it, namely, the operator a/b, which changes b into a. The laws of its operation depend on the kind of quantities represented by b and a. If b and a are ordinary numbers, the quotient is the ordinary fraction; if b and a are vectors, the quotient is a quaternion. What can be simpler in conception or more complete in statement? On the other hand, it is very questionable indeed if the profoundest meditation on

* The introduction of this phrase might easily suggest to the reader that Hamilton had erred in not sufficiently explaining his meaning. On the contrary, Hamilton's explanations are always full—almost prolix at times.

Drehstreckungen could ever have led the mind to the true conception of a quaternion, or to the powerful vector analysis which clusters round it.

However this may be, the authors, either in ignoration or in ignorance of what Hamilton has done, seem to think it necessary to try to "attach a precise meaning to Hamilton's definition," and they proceed to consider what relation connects Q, v, and V, when the Wendestreckungen v and V have their axes perpendicular to the axis of the Drehstreckung Q. They find

$$(\quad)vT = (\quad)QQV,$$

or symbolically, if Q' be written for Q^2 and v' for vT,

$$Q' = v'V^{-1}.$$

This, be it remembered, is a symbolic equation connecting *operators*, and not an equation connecting quantities. It is, of course, again an identity in quaternions. The assumed condition means that β and therefore α are perpendicular to $\mathrm{VU}q$, and hence

$$\beta q = \mathrm{K}q \,.\, \beta, \quad \alpha q = \mathrm{K}q \,.\, \alpha, \text{ etc.}$$

Hence, multiplying $\mathrm{K}q$ into both sides of

$$q\alpha\mathrm{K}q = \beta(\mathrm{T}q)^4,$$

we get

$$\alpha\mathrm{K}q = \mathrm{K}q.\beta(\mathrm{T}q)^2 = \beta q(\mathrm{T}q)^2,$$

and multiplying into q we have finally

$$\alpha = \beta q^2 = \beta q', \text{ say.}$$

The rotational equation then becomes

$$q'(\quad)\mathrm{K}q' = q^2(\quad)\mathrm{K}q^2 = \beta^{-1}\alpha(\quad)\mathrm{K}\alpha\mathrm{K}\beta^{-1}.*$$

Regarding their form of this equation, Klein and Sommerfeld say:

"*The quaternion Q' is represented as the quotient of two vectors v' and V, whose directions are perpendicular to the axis of Q' and make with one another an angle equal to half the rotation-angle of Q', and whose lengths are in the ratio of the tensor of Q' to unity.*

"This definition of quaternions is obviously somewhat par-

* In the absence of the tensor T the quaternion form is simpler than the Drehstreckung form.

ticular (*ziemlich partikulär*), and is inferior in simplicity to our original introduction of the conception. On the other hand, we must not conceal from ourselves that it has a great advantage over ours. In fact, it puts immediately in evidence the *half* angle of rotation ($\omega/2$) required for the unambiguous description of the quaternion, while our view of Drehstreckungen deals primarily with the whole angle of rotation (ω), and has then to be brought into relation with the half angle of rotation through the somewhat arbitrary rules of p. 36."

The "it" (*sie*) of the second sentence refers presumably to *Hamilton's* definition of a quaternion, although grammatically it refers to their own "somewhat particular" definition immediately preceding. This definition, however, is not Hamilton's in any strict mathematical sense. What follows in the paragraph just quoted, if taken in conjunction with foregoing statements, constitutes a remarkable confession. Hamilton's definition is first criticised as being "scarcely adapted to the end aimed at," but now it is admitted to have "a great advantage" over their view of a Drehstreckung, which, we are nevertheless assured, "leads to a complete, clear, and comprehensive conception of the quaternion calculus"; and one stated reason for this great advantage is that their "complete, clear, and comprehensive conception" has to be eked out by means of certain "arbitrary rules" regarding whole angles and half angles of rotation.

But, strictly and therefore mathematically speaking, their definition has to do, *not with a quaternion and two vectors*, but with a Drehstreckung and two Wendestreckungen, whose axes are subject to a particular limitation. A so-called quaternion Q' is represented as the quotient of two vectors v' and V; but with Q' Klein and Sommerfeld associate an angle of rotation double the magnitude of that which Hamilton would have called the angle of the quaternion v'/V.

In short they use Q', Q, v and V, each and all, in a double significance. When the exigencies of analysis demand it they simply follow Hamilton and Tait—that is, their analytical work is purely quaternionic. But when there is no direct question of establishing fundamental relations among the scalar quantities involved, they endow their so-called quaternion with powers that belong, as

Hamilton and Cayley showed long ago, to a particular quaternion operator. Because of its peculiar form this operator, viz., $q(\quad)q^{-1}$, involves the same four scalars which enter into the analytical expression for the quaternion q. These four scalars have long been known to be remarkably simple functions of the half angle of rotation and of the position of the axis of rotation symbolised by the operator $q(\quad)q^{-1}$. The modification introduced by Klein and Sommerfeld in their passage from the simple Drehung to the Drehstreckung is completely symbolised by the quaternion form $q(\quad)Kq$, a form already used by Tait (*Proceedings*, R.S.E., Vol. XIX., p. 196, 1892), while the equivalent form $uq(\quad)q^{-1}$, where u is a scalar multiplier (in fact Klein and Sommerfeld's *tensor* of the Drehstreckung), was used by Tait in his earlier paper on Orthogonal Isothermal Surfaces (*Transactions*, R.S.E., 1873–4; *Scientific Papers*, Vol. I., p. 180).

Thus, in their attempt to base the quaternion calculus on the conception of the Drehstreckung, the one novelty to be placed to Klein and Sommerfeld's credit is the identification of a quaternion with a very special kind of quaternion operator. Given the Hamiltonian quaternion q, it is a comparatively simple matter to pass to the required rotational operator $q(\quad)q^{-1}$. But to pass originally from the rotation to the quaternion with which it is *now known* to be so intimately associated would almost certainly have proved a feat beyond the powers of any mathematical mind. For what is there in the simple conception of a rotation to suggest the presence of a quantity or operator and its reciprocal?

The Examination of Sea-Water by an Optical Method.

By **J. J. Manley**, Magdalen College Laboratory, Oxford. *Communicated by* **Sir John Murray**, K.C.B.

(Read January 8, 1900).

In a paper* communicated to the Royal Society, Mr R. T. Günther and the author gave an account of the results obtained from the examination of two samples of water taken from Lake Urmi, and amongst other determinations of a chemical and physical nature, were those of the refractive indices, which were performed with the aid of the Royal Society's large quartz prism and spectrometer, the latter reading by means of micrometers to 2″ of arc. On comparing the values obtained for the refractive indices of the two samples of water with those obtained for the relative densities, it was at once apparent that the former differentiated the two samples quite as distinctly as the latter.

Krummel† attempted an optical method for the examination of various samples of sea-water, by measuring their refractive indices with the aid of an Abbé refractometer. The chief objections to the use of this instrument are—(1) Its sensibility is not sufficient when the waters to be examined differ but slightly from each other in their degrees of salinity; (2) the drop of water placed upon the fixed prism must necessarily undergo a certain although small amount of evaporation before it can be covered by the second or movable prism; (3) there is a considerable degree of uncertainty as to the true temperature of the liquid contained between the prisms, even when the refractometer is supplied with a water jacket. The thermometer indicates the temperature of the water in the jacket, but, owing to the unavoidable massiveness of the prisms, and the bad conducting power of glass for heat, it is highly improbable that the temperature observed is also that of the liquid whose refractive index is being measured.

* *Proc. Roy. Soc.*, vol. 65, 1899, p. 312.
† *Annalen der Hydrographie*, 1894, p. 241.

The Relative Densities.

In order to determine how far the optical method proposed by Krummel might be relied upon, Mr H. N. Dickson very kindly supplied the author with five samples of sea-water marked 1^v, 2^v, 3^v, 4^v, and 5^v, which differed but little from each other as regards "total salinity." The samples were first examined as follows:— Using a Sprengel tube having a capacity of about 48 c.c., two series of determinations of the relative densities at 24° C. were made. The tube was first washed out with fuming nitric acid, then with distilled water, and finally with absolute alcohol; it was then dried by keeping it thoroughly heated whilst a current of air was passed through; when the tube had become quite cold, it was wiped and hung from one arm of the balance, and after an interval of five minutes its weight was determined. The tube was then charged with recently re-distilled water, and suspended centrally in a large water bath, furnished with a rocking stirrer which was kept moving by a small water motor; the temperature of the bath was indicated by a standardised thermometer reading to 0°·1 C. With this apparatus the maintenance of a constant temperature, which differed very little from that of the room, was an extremely easy matter, the momentary application of a small Bunsen flame from time to time being all that was necessary. It was observed that the tube, together with its contents, assumed an almost constant temperature in about ten minutes after immersion in the bath; an approximate adjustment of the contents was then made. In every case, however, the tube was allowed to remain in the bath for twenty minutes, when the liquid was finally adjusted in the usual manner by the application of bibulous paper to the capillary. The tube was then removed from the bath, carefully wiped, again suspended from one arm of the balance, and weighed after five minutes. The contents were then discharged, the tube repeatedly washed out with portions of the sea-water to be examined, and then filled with it, and the process described above, repeated. After the first series of determinations had been completed, the tube was again thoroughly cleaned, dried and weighed, and a second series of determinations proceeded with in a manner identical with that described for the first series. The weighings were performed with a delicate long-

beam Oertling balance and a recently standardised box of weights. Table A shows the values obtained for the different weighings in the two series.

TABLE A.

Series.	Weight of tube.	Weight of tube + distilled water.	Weight of distilled water.	Water required to fill tube at 24° C.					
				1v.	2v.	3v.	4v.	5v.	
I.	16·8912	64·7332	47·8420	49 0754	49·0613	49·0561	49·0582	49·0738	Grams.
II.	16·8908	64·7302	47·8394	49 0770	49·0607	49·0585	49·0630	49·0752	

If W be the weight of a certain volume of sea-water which fills the Sprengel tube at 24° C. and w_1. the weight of the same volume of distilled water, also at 24° C., then W/w_1 expresses the relative density at the temperature named. The values shown in Table B. were obtained in this manner.

TABLE B.

Sample of Water.	Series I.	Series II.	Means.
1v	1·02578	1·02587	1·02582
2v	49	53	51
3v	38	48	43
4v	42	58	50
5v	75	83	79

The Optical Measurements.

The refractive indices of the five samples of water, together with that of recently re-distilled water, were next determined with the aid of the above-mentioned large spectrometer and hollow quartz prism. Two series of measurements were made at the ordinary temperature of the room, on two different days. The bottles containing the waters to be examined were placed upon a shelf, close to the spectrometer, the day before any measurements were proceeded with ; on the day of examination the water would

therefore be at almost, if not quite, the same temperature as that of the room itself, and any change in the temperature of the liquid, would be due chiefly to the slight though unavoidable handling, and to the proximity of the observer. The actual temperature of the water undergoing examination in the prism, was found by the standardised thermometer, which was used in connection with the density determinations already described, the reading being taken immediately after the position of minimum deviation had been found. The point of intersection of the cross threads in the telescope was in every case made to coincide with the right-hand edge of the image of the slit of the collimator, as it was found that far more concordant and trustworthy readings were obtained in this way, than by bringing the point of intersection upon the *estimated* centre of the image; successive readings of the same quantity, when effected by the latter method, were sometimes found to differ by as much as 6″ or 8″ of arc, whereas by the former, or edge-of-slit method, the various readings rarely differed by more than 2″, and were generally identical.

To find the value for the minimum deviation of the D line by a sample of water, the following method of procedure was adopted. (1) The prism was washed out twice with portions of the water to be examined, then filled, and the thermometer inserted; (2) the direct reading for the edge of the slit was made; (3) the prism was placed upon the spectrometer, and the position of minimum deviation found; (4) the temperature of the water in the prism was noted; (5) the position of minimum deviation was read; and (6) the prism was removed from the spectrometer, and the direct reading for the edge of the slit again taken. If the direct readings (2 and 6) differed by more than 2″ of arc, the whole process was repeated; this, however, was only found to be necessary in one instance. With a little practice the whole of the above operations may be performed in five minutes.

Temperature Corrections.

Before the values obtained for the minimum deviations or refractive indices could be compared with each other, it was necessary to study the effect produced upon them by a change in tem-

perature; for this purpose, the waters denoted by 1v and 5v were selected. The water having been introduced into the prism, the minimum deviation for the D line was found in the manner already described; the thermometer was then removed, and the aperture in the prism closed with a stopper. A Bunsen burner was then lit in the closed room, in order to raise the temperature; after a time, the minimum deviation was again determined, the process being repeated for two other and still higher temperatures. In this manner were obtained the minimum deviations at four different temperatures, as shown in Table C.

TABLE C. Water 1v.

	a.	*b.*	*c.*	*d.*
Temp. at which the min. dev. was taken.	20°·0 C.	22°·9 C.	23°·9 C.	24°·9 C.
Minimum deviation in secs. of arc.	86,884″	86,796″	86,766″	86,732″

The data given in the above table enable us to determine the value of a correcting factor, which may then be used to reduce all observed minimum deviations to a common temperature. In the fourth column of Table D, the values deduced for the factor are shown; it will be seen that the mean value is 31″.

TABLE D.

	Temp. diff.	Diff. in dev.	Diff. in the min. dev. for 1° C.
From experiments *a* and *b*	2°·9 C.	88″	30″·3
,, ,, *a* and *c*	3°·9 ,,	118″	30″·2
,, ,, *a* and *d*	4°·9 ,,	152″	31″·0
,, ,, *b* and *c*	1°·0 ,,	30″	30″·0
,, ,, *b* and *d*	2°·0 ,,	64″	32″·0
,, ,, *c* and *d*	1°·0 ,,	34″	34″·0
			MEAN = 31″.

Similar measurements applied to the water marked 5v led to exactly the same value for the correcting factor. Since the waters differ but slightly from each other, it may be assumed, without the

introduction of any perceptible error, that for all the five samples, the value for the minimum deviation of the D line diminishes by 31″ for an increase in temperature of 1° C. The following Table E gives the observed and reduced values for the minimum deviation for the several waters.

TABLE E.

Series.	Water.	Observed deviations in secs. of arc.	Temp. at which observation was made.	Deviations reduced to 24° C.
I.	1v	86,796″	22°·9 C.	86,761″
	2v	786	22°·7 ,,	744
	3v	774	22°·7 ,,	734
	4v	785	22°·5 ,,	737
	5v	800	22°·3 ,,	746
II.	1v	86,804″	22°·8 ,,	86,766″
	2v	768	23°·1 ,,	739
	3v	754	23°·3 ,,	732
	4v	760	23°·5 ,,	744
	5v	772	23°·6 ,,	759

From this table it will be seen that the means of the reduced minimum deviations obtained from series I. and II. are—

For 1v 86,764 seconds of arc at 24° C.
,, 2v 742 ,, ,, ,,
,, 3v 733 ,, ,, ,,
,, 4v 741 ,, ,, ,,
,, 5v 753 ,, ,, ,,

Discussion.

When investigating similar kinds of water by the optical method, we may express the differences observed in various ways; but for our present purpose it will be sufficient if we consider two only.

(*a*) We may select a prism having a strictly constant refracting angle of say 60°, and determine the values for the minimum deviations, δ, d_1, d_2, etc., of the D line for recently re-distilled water, and the waters under examination, a standard temperature being

maintained throughout; or if the temperature is unavoidably variable, proper corrections determined from time to time, must be applied in order to reduce all the observed deviations to a common temperature. The ratios $\frac{d_1}{\delta}$, $\frac{d_2}{\delta}$, etc., may then be obtained and compared. This method is analogous to a determination of the relative density of a liquid at some standard temperature, and may be termed the *relative deviation.*

(*b*) The refractive angle of the prism may be determined in addition to the minimum deviations of the D line by the waters, and from these data, the refractive indices, μ, μ_1, μ_2, etc., for recently re-distilled water and the waters to be compared may be calculated.

By the first or relative deviation method, a number of samples of water can be examined far more quickly than by the relative density method which is so generally adopted; this is due to the ease and rapidity with which a minimum deviation observation may be made, and to the fact that the calculation is of the same simple order as that used for obtaining the relative densities.

Let S_D and W_D represent the minimum deviations of the D line by sea-water and re-distilled water respectively, then the ratio $\frac{S_D}{W_D}$ gives the relative deviation. Applying this method to the waters under examination, and expressing the several minimum deviations in seconds of arc, we obtain the values shown in Table F.

TABLE F.

Min. deviation W_D, for re-distilled water at 24° C. = 85,018″.					
Water.	1ᵛ.	2ᵛ.	3ᵛ.	4ᵛ.	5ᵛ.
Min. deviation S_D at 24° C.	86,764″	86,742″	86,733″	86,741″	86,753″
Ratio $\frac{S_D}{W_D}$	1·02054	1·02028	1·02017	1·02027	1·02041

If we now compare the relative deviations given in this table with the relative densities given in Table B, we at once observe that the differences in the former are of practically the same magnitude as those exhibited in the latter; therefore, if we proceed to arrange the waters according to their degrees of total salinity or "total salts" per kilogram, the value obtained for the several relative deviations will enable us to differentiate the waters as sharply and decisively as the corresponding values for the relative densities, and the use of either method would lead us to arrange the waters according to the following descending order of salinity: 1^v, 5^v, 2^v, 4^v, 3^v. The waters 2^v and 4^v are practically identical, as both methods place the former in the higher position by only ·00001.

The second or refractive index method for comparing waters, requires, in addition to the minimum deviation observation, an accurate determination of the refracting angle of the prism: when these are known, the refractive index μ may be calculated from the well-known formula

$$\mu = \frac{\sin \frac{1}{2}(A + D)}{\sin \frac{1}{2} A.}$$

A being the angle of the prism and D the minimum deviation. This method would, however, probably prove to be far less convenient in practice than the relative deviation method; and since the value for μ increases or decreases with the deviation, one would be led to adopt the simpler or relative deviation method, rather than the other. It should also be observed that when the refractive indices for similar samples of average sea-water are compared with one another, the "total salinity" of one water is generally distinguished from that of another by a change in the value of the 5th decimal figure only; occasionally the 4th figure changes by 1; the relative deviations, on the other hand, may and do show well-marked differences in the 4th decimal.

In Table G the relative densities, relative deviations, and refractive indices of the five samples of water examined are grouped together, so that the results obtained by the different methods under consideration may be conveniently inspected and compared.

TABLE G.

Water.	Relative Deviations.	Diffs.	Relative Densities.	Diffs.	Refractive Indices, μ.
1^v	1.02054		1·02582		1·33882
		·00013		·00003	
5^v	41		79		78
		·00013		·00028	
2^v	28		51		75
		·00001		·00001	
4^v	27		50		75
		·00010		·00007	
3^v	17		43		71
	Mean Diff. = ·00009		Mean Diff. = ·00010		

The costliness of the refractometer which has been employed for the measurements detailed above, might possibly incline an individual observer to choose the usual specific gravity method rather than the one advocated here; but where a large number of samples of water have to be examined (as, for instance, in a central laboratory), the optical method would undoubtedly prove to be the most economical and convenient one, on account of the rapidity with which the determinations could be effected.

The author hopes that in a future communication he may be able to give an account of some further investigations which he intends to carry out with a special form of refractometer, which has been designed for studying the changes which the refractive indices of liquids undergo with change of temperature.

Further Investigations on the Life-History of the Salmon in Fresh Water. By D. Noël Paton, M.D., F.R.C.P.Ed., and M. I. Newbigin, D.Sc.

(Read December 4, 1899.)

(*From the Laboratory of the Royal College of Physicians of Edinburgh.*)

A. Further Evidence on the Factors determining the Migration of Salmon from Sea to River.

In the "Report on Investigations into the Life-History of the Salmon in Fresh Water," published in 1898, the changes which the fish undergoes between the months of May and November were dealt with, but there was no material available to enable the observations to be extended throughout the remaining five months of the year, from December to April.

The difficulty of getting an adequate supply of fish during these close months is very great, but through the energetic co-operation of Mr Archer and his successor in the post of Inspector of Salmon Fisheries, Mr Calderwood, a certain number of fish have been procured during these months from the estuaries of the Spey and the Dee.

To the Duke of Richmond and Gordon, through his commissioner, George Muirhead, Esq., and to the District Fishery Board (Aberdeenshire) of the River Dee, our thanks are due for generously supplying us with material.

In spite of the earnest endeavours of Mr Archer and Mr Calderwood, it has been found impossible to get "clean"—unspawned—fish from the upper waters during these months.

The methods employed in the present investigation were those described in our previous Report, pp. 3 to 7; and in comparing fish of different sizes with one another, all weighings are expressed as for fish of uniform size—100 cm. in length—called the standard fish, S. F. Weights are given in grammes.

The following Tables give the results of the examinations and analyses of twelve female fish taken in the estuaries during February, March, and April.

Although the amount of fats was determined in every case, it has not been considered necessary to give the results of these analyses apart from the analyses of the total solids.

TABLE I.—*Showing Length, Weight, Weight of Muscles and Ovaries per Fish and per Fish of Standard Length in Female Salmon from Estuaries.*

No.	Length.	Weight.		Weight of Muscle.		Weight of Ovaries.	
		Actual.	Per S. F.	Actual.	Per S. F.	Actual.	Per S. F.
			February.				
2	66	2680	9338	1680	5853	21	73
3	63	2370	9480	1494	5976	27	101
4	71	3490	9470	2230	6229	21	58
Average,	...	...	9429	...	6019	...	77
			March.				
5	73	4095	10500	2654	6795	36	92
7	67	3485	11578	2140	7109	25	83
8	70	3800	11078	2350	6852	21	61
10	66	2710	9442	1638	5707	21	73
35	67	3397	11323	1676	5576	29	90
Average,	...	...	10785	...	6408	...	80
			April.				
11	70	4070	11866	2560	7463	52	151·6
12	70	4170	12157	2700	7842	36	104·9
13	74	4350	10741	2844	7020	32	79·0
14	75	4680	11090	2970	7038	37	87·6
Average,	...	...	11463	...	7341	...	106

TABLE II.—*Showing Percentage and Total Amounts of Solids in Muscles and Ovaries in Female Fish from Estuaries.*

No.	Muscles.			Ovaries.	
	Per Cent.		Total per S. F.	Per Cent.	Total per S. F.
	Thick.	Thin.			
			February.		
2	30·6	35·0	1850	27·6	20
3	27·7	31·3	2443	31·1	36
4	34·4	39·0	2349	28·0	16
Average,	...	...	2214	...	24
			March.		
5	32·2	36·6	2010	31·4	28
7	32·4	38·0	2372	27·9	23
8	35·9	36·6	2471	24·2	15
10	31·6	34·4	1843	31·0	22
35	34·0	36·6	2568	32·0	31
Average,	...	...	2355	...	24
			April.		
11	32·3	39·4	2542	32·7	49
12	32·3	35·7	2657	31·6	32
13	32·8	38·0	2456	30·7	25
14	35·5	37·8	2741	30·5	28
Average,	...	...	2599	...	33

If the results of these investigations on the solids of salmon leaving the sea during February, March, and April, are compared with the results previously obtained during the other months of the year, the following table may be constructed.

TABLE III.—*Showing the Amount of Solids in Muscles and Ovaries of Female Salmon leaving the Sea throughout the Year.*

	Nov.	Feb.	Mar.	Apr.	May and June.	July and Aug.	Oct. and Nov.	Kelts.
Muscles, . .	2481	2214	2355	2599	2210	2270	1750	946
Ovaries, . .	23	24	24	33	47	72	545	9
Total, .	2504	2238	2379	2632	2257	2342	2295	955

Such a table fully confirms the conclusion previously arrived at—THAT THE SALMON GOES TO THE SEA TO FEED AND RETURNS TO THE RIVER WHEN IT HAS ACCUMULATED ITS FULL STORE OF NOURISHMENT IRRESPECTIVE OF THE CONDITION OF THE REPRODUCTIVE ORGANS. THE FACTOR DETERMINING MIGRATION FROM SEA TO RIVER IS NOT THE *nisus generativus*, BUT THE STATE OF NUTRITION.

B. MALE SALMON.

The number of male salmon examined in the course of the previous investigation was so small that it was considered unsafe to form any definite conclusions.

During the past two years every effort has been made to procure a supply of male fish, but without much success. The very small number of males which have been procured seems to indicate that they must be greatly outnumbered by female fish.

The following tables give the results of our examinations and analyses of the male salmon sent to us.

TABLE IV.—*Showing Length, Weight, Weight of Muscles and Testes per Fish and per Fish of Standard Length in Male Salmon.*

No.	Length.	Weight.		Weight of Muscles.		Weight of Testes.	
		Actual.	Per S. F.	Actual.	Per S. F.	Actual.	Per S. F.
ESTUARY.							
January.							
29	67	3112	10338	1992	6285	4	13·2
30	66	2922	10181	1900	6620	5	17·4
Average,	...	...	10259	...	6452	...	15·3
March.							
6	75	4410	10450	2784	6597	5	11·6
33	73	3840	9846	2508	6379	4	10·2
34	67	3090	10266	1900	6312	3	9·9
Average,	...	...	10167	...	6429	...	10·5
June.							
20	74	4245	10481	2652	6548	7	17·2
July.							
25	68	3335	10621	1980	6306	7	22·6
UPPER WATERS.							
June.							
21	74	3755	9270	1660	4100	27	60
22	69	3200	9756	1410	4300	15	457
24	74	3815	9420	1643	4057	53	131
Average,	...	...	9482	...	4152	...	78

TABLE V.—*Showing Percentage and Total Amount of Solids in Muscles and Testes of Male Fish.*

No.	Muscles. Per cent. Thick.	Muscles. Per cent. Thin.	Muscles. Total per S. F.	Testes. Per Cent.	Testes. Total per S. F.
ESTUARY.					
A. *January.*					
29	30·2	33·4	2546	16·3	2·15
30	31·4	33·9	1962	19·2	3·06
Average,	...	...	2254	...	2·60
B. *March.*					
6	31·4	35·4	2122	19·4	2·33
33	32·8	37·6	2200	19·5	1·96
34	32·6	36·0	2132	13·6	1·34
Average,	...	...	2151	...	1·87
C. *June.*					
20	31·5	36·8	2177	15·4	2·58
D. *July.*					
25	34·0	40·0	2238	18·2	3·09
UPPER WATERS.					
June.					
21	30·0	31·7	1686	16·7	10·3
22	31·8	34·6	1845	16·2	7·3
24	28·6	30·9	1596	16·4	21·7
Average,	...	...	1711	...	19·6

Comparison of the results of the present investigation with those recorded in the previous Report tend to show that the male fish leaving the sea from January to August have all about the same amount of solids in their muscles and have testes little developed.

TABLE VI.—*Showing Solids of Muscles and Testes of Male Salmon leaving the Sea.*

	Jan.	March.	May and June.	July and Aug.	Oct. and Nov.
Muscles, .	2254	2151	2004	2345	1470
Testes, . .	2·6	1·9	2·6	3·9	66
	2256·6	2152·9	2006·6	2348·9	1536

The slightly lower figure in May and June is due to the fact that the two fish examined in 1896 were much below the average as regards muscular development.

The two fish examined in October and November show a very small amount of solids in the muscles. The average figure for the total solids from January to August—2191 grms.—is based on the examination of 11 fish, and the divergence from this manifested in these two fish must be accepted with caution, and does not justify the formation of any conclusions. Further data are required.

From the table given above it will be seen that the male salmon coming from the sea closely resemble the female fish in the amount of nourishment stored in the body.

Amount of solids in muscles and genitals in salmon leaving the sea from January to August:

Female Fish,	2434
Male Fish,	2191

In fact, the more extended examination of these male fish from the estuaries, further bears out the conclusion, arrived at from the examination of female fish, as to the factors determining migration.

Comparing the upper water male fish taken in 1898 with those taken in 1896, it is seen that the June fish in the former group resemble the July and August rather than the June fish in the latter group. What the explanation of this may be is not manifest. Possibly an earlier migration to the river may have induced an earlier development of the testes and a greater loss of substance from the muscles.

C. On the Nature of the Phosphorus Compounds of the Muscles of Salmon, and the Synthesis of the Organic Phosphorus Compounds of Testes and Ovaries.

From the study of the phosphorus compounds in the muscles and in the testes and ovaries at various seasons (Report, p. 143 *et seq.*), we came to the conclusion that the nucleic acid in the testes and the ichthulin in the ovaries—both complex organic phosphorus compounds—are built up from simple inorganic phosphates stored in the muscles.

The recent researches carried on in Röhman's laboratory (*Berl. klin. Wochensch.*, 1898, p. 789) tend to show that, in dogs at least, inorganic phosphorus compounds are not used in the body to anything like the same extent as organic compounds; and the fact that in our previous investigation we assumed all the phosphorus extracted by acidulated water to be inorganic in nature, rendered it necessary to make further observations. Especially was this the case since Siegfried (*Ztsch. f. phys. Chem.*, Bd. xxi., p. 360, 1896) has shown that in the flesh of mammals some of the phosphorus thus extracted is in organic combination, being linked to a substance which he has described as carnic acid. He states that carnic acid has the formula $C_{10}H_{15}N_{3}O_{5}$, and that it is identical with antipeptone.

If this is so, the phosphorus compound—which he calls phosphocarnic acid—must be nearly allied to the pseudo-nucleins. If such a body occurs in the muscle of the salmon in sufficient quantity to yield the phosphorus of the nucleic acid of the testes and of the ichthulin and lecithin of the ovaries, the conclusion as to the extent of synthesis may have been erroneous.

In the previous Report it was shown that the average amount of

phosphorus in the muscle of the salmon is 0·215 per cent., and that of this about 0·109 is soluble in water.

To determine how much of this is in organic combination and how much in such compounds as phosphocarnic acid, the following observations were made:—

1. 100 grm. of the flesh of a fresh sea salmon in March 1899 were extracted repeatedly with over 2 litres of water and acetic acid. The watery extract was boiled and the precipitate well washed. The inorganic phosphorus was precipitated with ammonia and chloride of calcium. In the precipitate the phosphorus was estimated in the usual way, calcium being washed out of the molybdate precipitate with 10 per cent. nitrate of ammonia solution.

$$Mg_2P_2O_7 = \cdot357, \ P_2O_5 = \cdot228, \ P = \cdot099$$

The filtrate containing any phosphocarnic acid was evaporated, burned and treated with molybdate of ammonia, and P. estimated as above.

$$Mg_2P_2O_7 = \cdot017, \ P_2O_5 = \cdot011, \ P = 0\cdot005$$

2. 135 grm. of the flesh of a kelt (32) captured in March, was analysed in same way:—

$$Mg_2P_2O_7 = \cdot330, \ P_2O_5 = \cdot211, \ P_2O_5\% = 156, \ P = 0\cdot068\%$$

The filtrate, containing any phosphocarnic treated as above, gave no precipitate with ammonia-magnesia mixture.

The phosphorus extracted by water from the muscle is almost entirely in simple inorganic combination.

The evidence thus supports the view that the ovarian ichthulin and the testicular nuclein are built up from simple inorganic phosphorus compounds derived from the muscle.

D. Further Observations on the Source of the Pigment of Salmon Muscle.

By M. I. Newbigin, D.Sc.

On the Pigments of certain Crustacea.

It is well known that the salmon when in the sea feeds largely on herring, and that these in turn prey upon small free swimming crustacea, many of which have a bright red colour. It therefore

seemed of interest to compare the pigment of such crustacea with the colouring matter of the muscles and ovaries of the salmon.

During last summer, Sir John Murray sent to the Laboratory of the Royal College of Physicians, collections of crustacea obtained by tow-netting in Loch Fyne, in order that the pigments might be investigated. The crustacea sent were all of a red colour, and are believed to constitute the chief food of the herring. The object of the investigation was to find what relation, if any, the pigments of these crustacea bear to those of the salmon.

When received the crustacea were preserved in methylated spirit or in alcohol of various strengths. In no case was the preserving fluid markedly coloured, most of the pigment being still retained by the organisms. As to the crustacea sent, there were separate bottles of *Pandalus annulicornis* and *Hippolyte scurifrons,* and also large bottles labelled "contents of tow-net in upper Loch Fyne." These last contained chiefly copepoda intermixed with colourless organisms such as *Sagitta* and also various Euphausidæ, species of *Hippolyte,* etc. The larger crustacea were picked out from among the copepoda, and the pigments investigated in two sets—(1) those of the copepoda, (2) these of the other crustacea.

1. The copepoda contained a large amount of fat in which the pigment was dissolved. It was found possible by squeezing to extract from their bodies drops of fat deeply coloured by the reddish pigment. Both fat and pigment dissolve in boiling methylated spirit; but on cooling, the coloured fat separates out at the bottom of the vessel. Both fat and pigment dissolve readily in ether, which is thus a much better solvent for the pigment than alcohol. When the fat is saponified either by heating with an alcoholic solution of caustic soda, or by adding metallic sodium to a solution in ether, a red soap is formed from which the pigment may be obtained after treatment with acid. A small amount of a yellow pigment remains in solution in the caustic solution after saponification, as in the case of the salmon pigment.

The red pigment is a lipochrome, and exhibits the same general characters as the red pigment of the salmon, but it was not obtained in sufficient amount for detailed investigation. It especially recalls the pigment of the salmon in its close association with fat.

2. The pigments of the other crustacea sent closely resembled those of the Norway lobster. The most distinct difference from the copepoda lies in the fact that the red pigment is chiefly found in the chitinous cuticle and in the epidermis; the occurrence of a coloured oil was not obvious.

On the Rectal Gland of the Elasmobranchs. By J. Crawford, M.B., C.M. *Communicated by* Dr Noël Paton. (With a Plate.)

(Read December 4, 1899.)

(*From the Laboratory of the Royal College of Physicians of Edinburgh.*)

The so-called rectal gland of Elasmobranch fishes claimed notice early in the history of scientific research, as might indeed be expected from the obviousness of its appearance and the invariability of its occurrence. But, in spite of this fact, the rectal gland remains one of those organs the knowledge of the structure of which is unsatisfactory, and the conjecture as to the function of which is consequently hazy.

As far as can be ascertained, Professor Monro of Edinburgh (the second of that well-known name) gave the earliest description of the rectal gland in his work upon the *Structure and Physiology of Fishes*, published in 1785. He refers to the organ as the "appendix digitiformis," the "appendix vermiformis," and in one place as the "cæcum." Duméril (4), in *Suites à Buffon*, amplified Monro's description, and various other writers on zoology have taken the subject into consideration, the latest structural description being that of Blanchard, published in 1880 (7).

I have been unable to find any description of the microscopical appearances of the structure of specimens prepared by the later and more satisfactory methods of investigation, and it seemed therefore of interest to make an attempt to elucidate the structure by a study of sections prepared by such methods. I shall accordingly first give a short account of the characteristics observed, and afterwards consider briefly the various theories which have been advanced regarding the possible function of the organ.

Macroscopic Appearances.

To the naked eye the rectal gland presents an appearance varying somewhat in the different genera in which it occurs, the

principal difference being that in Batoids the duct which leads from the gland proper to the rectum is short and comparatively wide, and opens into a posterior dilatation of the rectum, while in Selachians the duct is longer and narrower, and opens more directly into the rectum (Howes). The opening of the duct, which is usually guarded by a fold of mucous membrane, is upon the dorsal wall of the rectum, about midway between the anus and the termination of the spiral valve. The gland is connected to the posterior abdominal wall by a fold of peritoneum, and is, according to Howes, supplied by the superior mesenteric artery. Its size varies in proportion to that of the animal, and in a skate of two feet in length from tip of snout to tip of tail may be about an inch in length; its colour is usually a reddish-brown. A longitudinal section shows that there is a central smooth-walled canal, irregular in calibre and giving off numerous short branches, surrounded by a firm glandular tissue, which is in its turn compassed by a whitish ring of fibrous tissue; surrounding all is the coat of peritoneum. Along the lumen can be seen the mouths of severed vessels, and it contains some dirty-yellow secretion of a viscid consistence and a neutral reaction. Practically nothing more can be discovered by the unaided eye.

Microscopic Appearances.

For the microscopic examination of the organ, specimens obtained in the freshest possible state were hardened in corrosive sublimate, formal in 8 per cent. solution, and in alcohol; and though good results were obtained by each method, it was noticed that the gland cells seemed to be best preserved by the alcohol; there was, however, considerable shrinkage. The specimens were then embedded in the usual way in paraffin, cut by a rocking microtome, and stained either by hæmatein and eosin, Ehrlich-Biondi triple stain, picro-carmin, or methyl-blue. Heidenhain's iron and hæmatoxylin method was also tried.

The organ may be described as consisting of three regions:

(1) An outer fibro-muscular layer covered by peritoneum.

(2) A middle glandular layer.

(3) A central region, consisting of ducts and blood-vessels arranged round a central lumen.

1. The outer layer, on which appears externally a coating of somewhat cubical peritoneal cells, is made up of bands of white fibrous tissue interwoven irregularly with a considerable amount of non-striated muscle-fibre running in a circular and longitudinal direction. In this tissue are to be found at intervals large sinuses of an irregular shape, lined with endothelial cells, and containing blood-corpuscles. Towards the inner part the muscle-fibres become closer, forming a definite band resembling a muscularis mucosæ external to the glandular tissue of the middle layer. Under a high power there is nothing further to be remarked.

2. The middle or glandular layer is composed of a number of few-branched tubules, radially arranged, separated by capillaries, which are usually gorged with blood. Under a high power the gland cells are seen to be cubical, mono-nucleated, ill-defined from one another, and of a granular appearance. This latter is due to the protoplasmic network and not to the presence of any foreign substance. The iron and hæmatoxylin staining method recommended by Heidenhain for showing zymin granules gave here a negative result. The nuclei of the cells are large, possessing an evident nuclear membrane and nuclear network, and showing three to five nucleoli. The fibrous tissue of this layer is slight, consisting of thin septa passing inwards from the outer fibro-muscular layer first described.

3. The central layer begins at a varying distance from the periphery by the sudden transition of the gland cells into the epithelium of ducts, which open after a short course into the central lumen.

Between these ducts and immediately external to the epithelium of the lumen are seen very large irregularly-shaped sinuses lined with endothelial cells, and filled with blood-corpuscles.

The lumen of the organ is large, though it is often compressed so as to seem almost valvular; in many cases it contains a granular substance of indefinite structure, which was unaffected by the staining reagents employed.

On examining the layer under a high power, the epithelium of the ducts and lumen is seen to be of the type described as transitional, showing several layers of polygonal cells, flattened as they approach the free surface. In many cases the more superficial

cells have undergone a mucoid change, and a band of clear cells is visible lining the duct. This appears to be a degeneration, since in other cases no such cells are to be seen.

The blood filling the sinuses was often remarked to contain many large and very coarsely granular eosinophilous cells.

The general system of the organ recalls, therefore, that of a compound tubular gland with short secondary ducts opening into a main central one. It might also be considered, more correctly from a developmental point of view, as a blind tube having the same general structure as that of the intestine, and presenting a lumen bounded by walls of a constitution comparable to that of the intestine, though widely differing from that part of it in the near neighbourhood.

The cells of the gland acini present no very peculiar feature; they resemble in general character the cells of the kidney, and suggest an excretory function rather than a secretory.

But what cannot fail to be noted in the structure of the organ is the richness of its blood-supply and the peculiar arrangement of that supply. There is a peripheral and a central arrangement of large blood-sinuses connected by a copious network of capillaries which bring the blood into intimate relation with the cells.

And though this has been remarked, attention seems never to have been arrested by the position of the central sinuses, directly in relation with the epithelium of the lumen; a condition which is surely uncommon.

Chemical.

Dr Noël Paton has been kind enough to make for me a chemical examination of the gland and its secretion, and has given me the following particulars:—The contained secretion of several rectal glands was preserved in absolute alcohol. The alcohol was evaporated off, and the residue extracted with water. A considerable amount of insoluble matter remained. The aqueous solution when treated with an alkaline solution of hypobromite of soda, gave a fine effervescence; and on the addition of oxalic acid as it evaporated, yielded a crop of crystals, some with the characteristic shape of oxalate of urea, some long and acicular. The secretion from the gland undoubtedly contains a considerable amount of urea.

Consideration of Function.

With reference to the probable function of the organ I have attempted to describe, several theories, more or less vague, have been presented. Monro, in the original notice, was of opinion that the organ was a secretory one, and Duméril calls it "a true secretory organ"; but neither offers any suggestion as to the probable nature of the supposed secretion. Leydig (5) compared its structure to that of the glands of Brunner in other animals, pointing out that in the genus Chimaera, in which a true rectal gland, as a separate viscus, does not exist, glandular tissue is present in the wall of the intestine at a corresponding situation, while in those fishes which possess a rectal gland the intestinal wall is in that region devoid of such tissue.

Home (2) compared the organ to the cæcal pouches of birds, and Retzius on that account suggested the title Bursa Fabricii.

Blanchard, while apparently demonstrating the hypoblastic origin of the organ, is of opinion that it is analogous to the anal or circumanal glands of some vertebrates, and prefers the name "glandula superanalis" to "rectal" or "digitiform."

But, as Howes points out, such an analogy is probably fallacious, since the circumanal glands are almost certainly derived from the epiblast.

Hyrtle, as quoted by Howes, supposed that the function was one accessory to reproduction, basing his belief upon a fancied increase in size of the organ in animals whose oviducts contained eggs, and upon his failure to detect food-stuffs within the organ. Howes could find no evidence to support Hyrtle's theory, and observes that the identity of the structure in each sex is a strong objection to it.

Howes, who notices and discusses these suggestions in an exhaustive paper, upholds the view that the function of the rectal gland is a secretory one, and concludes from this belief, and from its development and position, that it is to be compared to the vermiform appendix of other vertebrates. In the conclusion of his paper he writes:—"In the fact that the organ is a secretory one, we have, in the long run, a further point of agreement with the cæcum coli and appendix vermiformis. The fact that the

latter becomes adenoid in its most highly differentiated form, while the processus digitiformis is not known to be thus constituted, would appear to be of minor significance by analogy with Weldon's discovery that the suprarenal body in the Icthyopsida (Bdellostoma) probably represents a metamorphosed excretory blastema." The theory seems a plausible one, but as Howes nowhere refers to any actual work upon the structure of the organ, it is conceivable that he may not have thoroughly appreciated the distinct histological difference of the rectal gland from the vermiform appendix. The case of analogy he cites seems scarcely conclusive, and he seems to take for granted that the gland is secretory and not excretory, a view which is upheld by no direct evidence.

On taking a general view of these suggestions, none of them are entirely satisfactory. It seems unlikely that the gland is concerned in reproduction, as Hyrtle supposes. If, as Leydig thinks, it is of a nature resembling that of the glands of Brunner, its glycerin extract might be expected to show some digestive action.

The rich blood supply, the character of the secreting cells, resembling so closely as they do the cells of the kidney, and the occurrence of urea in considerable amount in the secretion, all point to the structure having an excretory function, and playing the part of a supplementary kidney.

When the peculiar richness of the blood and tissues of the elasmobranchs in urea is remembered, this action of the rectal gland becomes of very considerable interest.

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Mr J. Crawford on Rectal Gland of Elasmobranchs.

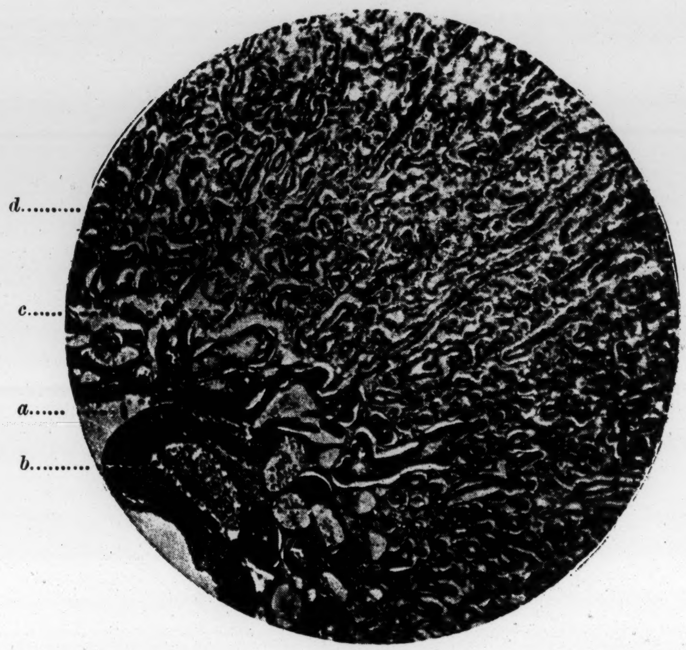


Fig. 1.

Fig. 2.—High-power view of gland acini.

DESCRIPTION OF PLATE.

Fig. 1.—Trans. section, rectal gland of skate. × Leitz object. No. 3. *a*, lumen; *b*, blood sinuses; *c*, ducts; *d*, Secreting Tubules.

Fig. 2.—The same. × Leitz $\frac{1}{12}''$ oil-immersion, to show Epithelium lining Secreting Tubules. High power view of gland acini.

A New Form of Myograph and its Uses.

By S. C. Mahalanobis, B.Sc., F.R.M.S., F.R.S.E., Assistant Lecturer on Physiology, University College, Cardiff.

(Read December 18, 1899.)

In connection with some investigations dealing with the velocity of muscular contraction under different conditions, I found it necessary to design a special apparatus for certain experiments. It subsequently occurred to me that, with some modification, this instrument could with advantage be used for various myographic purposes. I was thus induced to make the necessary additions and alterations—adapting the instrument for some special, as well as for most of the ordinary experiments in which a myograph is used.

A. *Description of the Apparatus.*

The instrument has a T-shaped lever (A) turning on a short axle passed through the centre of the head and so pivoted as to admit of free horizontal movements of the lever. To the long arm of the lever is attached a piece of straw provided with a writing point which records its history on a horizontal cylinder. The short arms have a number of holes into which S-shaped muscle hooks can be inserted. At a little distance from the support of the lever—at about the middle of the ebonite plate (G) that forms the base or floor of the instrument there is a fixed block of ebonite (C) forming a small support for two strips of brass (B) that are used as clamps. The two pieces of brass are insulated from one another—each being held on the top of the ebonite block by means of a pair of milled-head screws. Just behind the clamps there is a small upright rod (F) carrying a pair of electrodes (E) that can be held at any level. Still further back and near a corner there stands a firm pillar (H) supporting an electro-magnet (M) with adjustments for movements in two directions, *i.e.*, the electro-magnet can be raised or lowered and also moved backwards or

forwards as necessary. The armature (K) of the electro-magnet is hinged at the top, and its lower end—which is provided with a hook—can, when not held by the electro-magnet, freely swing forward. The small pulley (P) on the opposite side has a hole passing right through the centre of the support, so that a thread

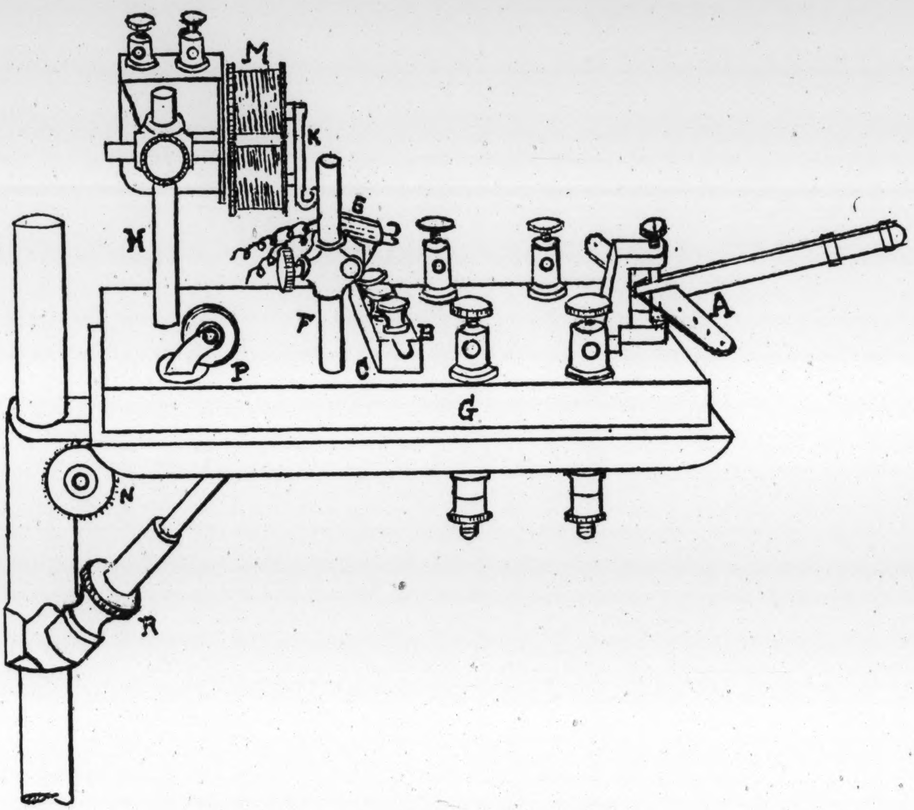


FIG. 1.—Side view of Myograph.

A, lever ; B, clamp ; C, ebonite block ; E, electrodes ; F, support for electrodes ; G, ebonite base ; H, support for electro-magnet ; M, electro-magnet ; K, armature ; P, pulley ; N, screw for clamping instrument on stand ; R, fine adjustment for bracket.

attached to the proximal end of the lever, stretching over the pulley can pass through this hole and suspend, below the instrument, a very light scale-pan carrying a small weight. The pulley being made on the principle of a caster, readily adapts its position in accordance with the movements of the lever. The four binding screws on the top of the ebonite plate are respectively connected

with four corresponding binding screws below the instrument; thus all necessary connections can be easily made even when the upper part is covered by a glass moist chamber. The instrument is supported by a strong brass bracket which can be firmly clamped on the stand by means of a screw (N), and is also provided with coarse and fine (R) adjustments for inclining it at a convenient angle.

B. Uses.

1. The chief purpose for which the instrument was designed was to obtain a method of graphic representation of the character and velocity of the contraction of frog's muscle immediately following an absolutely isometric stimulation. If a muscle is stimulated—say electrically—in the ordinary course of events, it contracts. But when a muscle is prevented from shortening during stimulation it undergoes a change of tension. This change of tension while the length of the muscle remains unaltered has been designated by Fick* as *isometric* condition. In my instrument a simple contrivance has been made for the rapid contraction of a muscle immediately after its tension is raised under absolute isometric condition, or in other words, by means of single induction shocks I have produced in frog's muscle a condition resembling what in the case of rapid voluntary contraction, has been called by Haycraft † "hold and let go" method.

For this purpose a nerve-muscle preparation of frog's gastrocnemius is supported horizontally, its femoral end being firmly clamped and the tendo Achillis fixed to the lever by means of a hook as indicated in fig. 2. On the other side of the pivot the lever is held by means of a very thin elastic band (O) clamped at one end like the muscle and attached to the lever at the other. Although the elastic band is able to hold the lever in position, keeping, on the other side, the muscle suspended without any laxity, it has only a small amount of *initial tension*; so that even when it is fully extended, due to the movement of the lever during contraction of the muscle, the elastic tension of the band does not

* *Arbeitleistung und Wärmeentwickelung bei der Muskelthätigkeit.* Leipzig, 1882, S. 131; also Pflüger's *Archiv*, Band xlv. p. 297.

† *Journal of Physiology*, vol. xxiii. Nos. 1 and 2.

exceed say five grams. A piece of string with a hook at each end connects the armature of the electro-magnet with the lever, on the same side as the elastic band; the string is of such length that, when the armature is held in contact by the electro-magnet, any contraction of the muscle immediately exerts a pulling force on it.

The muscle can be stimulated by its nerve placed on the platinum electrodes (E), or directly by sticking in two pieces of thin wire led off from the adjacent binding screws connected with the secondary coil.

The primary coil of an inductorium is so connected with the

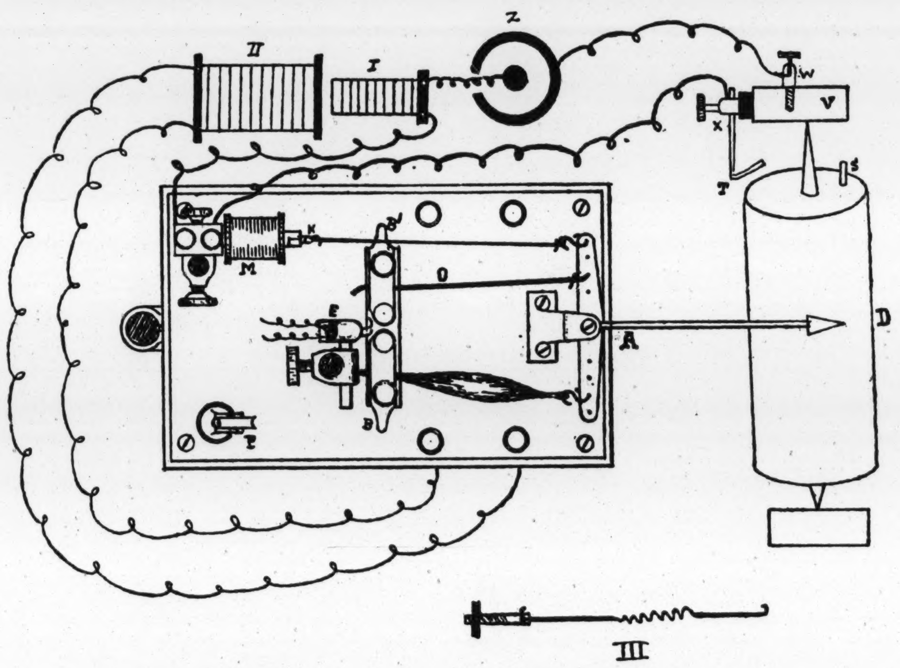


FIG. 2.—Surface view of Instrument and necessary connections.

A, lever; B B, clamps; D, drum; E, electrodes; M, electro-magnet; K, armature; O, elastic band; P, pulley; S, pin; T, spring; V, support of drum; Z, battery; I. primary coil; II. secondary coil; III. steel spring for isometric contraction.

battery as to include the electro-magnet of the myograph in the circuit. Instead of using an ordinary key a special device is made in which the revolving drum (D) is utilised for closing or opening the circuit. The binding screw W (fig. 2) is in contact with the metal support (V) of the drum, whereas a piece of ebonite insulates

the other binding screw (X) from the support. During the revolution of the drum the pin S touches the spring T and thus closes the circuit. The duration of the contact between the pin and the spring can be altered by adjusting the position of the spring, or in other words, the time-interval between the closing and the opening of the circuit can thus be regulated. The secondary coil is so adjusted as to obtain maximal stimulus both on making and breaking. Now, it is evident that during the revolution of the drum, as soon as the pin S touches the spring T, the circuit being completed, the muscle stimulated by the "make" shock tends to contract, but is prevented owing, at the same time, to the armature (K) being firmly held by the electro-magnet. Thus isometric condition of the muscle is attained. Quickly following this raising of the tension of the muscle the circuit is broken, and the "break" shock again stimulates the muscle, which rapidly contracts with freedom, the electro-magnet having now ceased to act.

Thus we are able to record on the smoked surface of the revolving drum the character and velocity of the contraction of the muscle under such modified condition. Detailed account of a series of observations will be published in a subsequent paper, but it may be briefly mentioned here that by this means a greater velocity of contraction is obtained and the rate of work is also increased.

2. *For isotonic contraction* the electro-magnet is thrown out of the primary circuit, and the lever is freed from the armature by taking out the hook attached to the string. The muscle is fixed as in the former case, and the end of the elastic band which is attached to the lever is brought very close to the pivot, so that during contraction of the muscle there will be very little extension of the elastic band. The muscle is stimulated by "make" or "break" shock—preferably the latter—using an ordinary key for this purpose, and the characteristic myogram is obtained on the smoked surface.

3. *For isometric contraction* the elastic band (O) is replaced by a steel (spiral) spring, fig. 2, III., one end of which is fastened to the brass clamp B′, and the other hooked on the lever so that the muscle, when stimulated, shortens against great resistance. By adjusting a screw connected with the spring the initial tension of

the muscle can be varied. The amount of tension at different stages of contraction of the muscle can be estimated by noting the extent of deviation of the writing point from the abscissa produced by known weights placed on a scale-pan suspended below the instrument by a string tied to the muscle hook and passed over the pulley (P).

The pulley can be similarly utilised in experiments on elasticity and extension of muscle, etc. Then besides most of the common experiments on the physiology of muscle, *e.g.*, fatigue, tetanus, etc., the instrument can with some manipulation be used to illustrate the action of antagonistic muscles by using a pair of gastrocnemii of the frog.

C. Advantages.

It seems to me that, apart from the special use of the apparatus, this form of myograph, with horizontal movements of the lever, has some advantage over the usual form where the lever moves in a more or less vertical manner. In the first place, here the influence of gravity on the movements of the lever is *nil.* Besides, in the case of vertically moving levers we find that, even when the lever is very light and the weight attached to it is small, the lever, owing to its mass and moving with great rapidity, gathers momentum: in virtue of which not only the lever tends to move upwards even when the contraction of the muscle has stopped, but also the tension of the muscle is diminished, thus seriously interfering with the isotonic condition of the latter. The same thing happens in the opposite direction during the downward excursion of the lever, *i.e.*, it continues to pull down the muscle beyond its initial extension. Thus the so-called *isotonic* curve is rendered untrustworthy, as has been strenuously urged by Kaiser.*

In a horizontally moving lever, where a very thin elastic band is used, and the point of its attachment is close to the fulcrum, the slight increase of tension of the elastic band, due to its extension during the contraction of the muscle, tends to neutralise the influence of the momentum of the lever.

* *Zeitschr. f. Biol.*, vol. 33.

The Presence of Enzymes in Normal and Pathological Tissues. By John Souttar M'Kendrick, M.D.*

(Read December 18, 1899.)

The unorganised ferments or enzymes which are present in the digestive juices have for many years occupied the attention of physiologists. Although their chemical nature is still doubtful, yet most of their physical and chemical characters are known, and there are methods by which they may be extracted from the tissues and digestive juices. They are generally believed to play the most important part in the digestive process, and within recent years physiologists and pathologists have speculated as to the existence of similar substances in other tissues, and so have endeavoured in many instances to offer a hypothetical explanation of some of the changes that occur in tissue cells themselves. During the last eighteen months I have endeavoured to ascertain the presence or absence of these enzymes in normal and pathological tissues generally. Before describing the method adopted in carrying out this research, with the enumeration of the tissues examined and the results obtained, I shall briefly refer to our present knowledge of the existence of these enzymes in tissues other than those of the digestive tract, as well as to their presence in plants.

Do Enzymes Exist in other Tissues?

Zymolysis, one of the manifestations of the digestive process, occurs in plants as well as in animals. We know from the researches of Bernard † that digestion in plants is in most cases an interstitial one. By that term he meant the chemical changes that take place in the food stored up in the tissues for purposes of nutrition. For example, the starch that exists in the tuber of the potato undergoes conversion into sugar at one period of its growth.

* This is an abstract of the original paper. The research was conducted partly in the Physiological Laboratory of the Glasgow University, and partly in a laboratory of my own at home.

† *Leçons sur les phénomènes de la vie*, T. 2, 1879, Paris.

Many other instances could be cited, which show that an interstitial digestion is being carried on in the cell structure of the plant, presumably by enzymes of a nature identical with those that exist in the digestive juices of the animal. The zymolysis then of plant life is the process of conversion of stored up food stuffs into new substances. These new substances are formed by the activity of the soluble unorganised ferments or enzymes. The zymolytic processes in plants have been investigated by Green,* Hansen,† Wortmann, and others, and it is now generally believed that in most plants there are at work enzymes of proteolytic, amylolytic, and inversive powers. The papaw plant contains a proteolytic enzyme, papain, which is very similar in its action to trypsin, and moreover the action of the enzyme compares favourably as regards activity with those of the proteolytic ferments of animal origin.

Again, it is generally admitted that the inversion of cane sugar (as, for example, beetroot sugar into inverted sugar), during the inflorescence of the plant, is due to an inversive enzyme. Many examples could be cited which show the presence in plants of enzymes similar in their nature and action to pepsin, ptyalin, and invertin.

The question presents itself—since we are aware that in plant life a zymolytic interstitial digestion is constantly at work—is it not possible, and indeed probable, that in animal tissues as well, enzymes are in action: they may be of the same or a different nature, taking an active part in the metabolic processes occurring in the individual cell? If such were the case, it might account for the conversion of glycogen into sugar in certain circumstances—the conversion depending upon the activity of a soluble enzyme, liberated, it might be, from a parent zymogen existing in the protoplasm of the hepatic cell. Again, it might account for the abnormal sprouting of a parent tissue, depending upon the increased activity of an enzyme in that tissue. When a sarcoma or a carcinoma grows, is it not possible that an interstitial digestion is at work, altering the nutrition of the parent tissue? This

* *Science Progress*, London, vol. i. p. 342 ; vol. ii. p. 109 ; vol. iii. pp. 68, 376 ; vol. v. p. 60.

† *Bot. Ztg.*, 1886, S. 137.

might account for the greater rapidity in growth of tumours in certain tissues than in others.

The only reference in literature which I have found bearing on this subject can be found in Halliburton * and Sheridan Lea's † books, which refer to the work of Nasse and Brücke more especially. Halliburton says, "Brücke has shown that muscle, in common with most of the tissues of the body, contains a small quantity of pepsin;" and again, "O. Nasse showed that muscle juice also contains an amylolytic ferment, which he supposes to act in the transformation of glycogen into sugar after death. I (Halliburton) have made a few experiments on this subject, and can fully confirm Nasse's statement of the existence of this ferment;" and again, he says, "We have already seen that such a ferment (diastatic ferment) can be obtained from muscle, and it seems that diastatic activity is present in all living proteids."

Sheridan Lea, when writing of ptyalin, states:—"While occurring chiefly and characteristically in saliva, a similar enzyme may be obtained in minute amount, but fairly constantly from almost any tissue or fluid of the body, more particularly in the case of the pig."

In an article by Brücke,‡ entitled "Beitrage zur Lehre von der Verdauung," there is a paragraph at the close entitled "Die verdauende Substanz im Fleische."

This is the subject evidently referred to by Halliburton, although Brücke may have described his results more fully in other papers. He showed that the juice of flesh when treated with water, and subjected to the same ether and cholesterin process that he used in carrying out his experiments for the isolation of pepsin from the mucous membrane of the stomach, had decided digestive properties. The digestion was noticeable in from five to six hours, and in the course of the next day all fibrin had been completely digested. He confirmed his results by a slightly different method. He obtained the juice from 4 lbs. of ox beef, and treated this with phosphate of lime. The filtrate was dissolved in weak hydrochloric acid. He obtained

* *Text-Book of Chemical Physiology and Pathology*, pp. 412 and 549.

‡ "The Chemical Basis of the Animal Body," Foster's *Physiology*, vol. v. p. 56.

‡ *Sitzung. Akad. der Wissensch.*, Band xliii. Abth. 2 (1861).

again a fluid which dissolved pieces of fibrin in the course of the same day. The digestion was found to go on, not only at 38° C., but even in an ordinary atmosphere. This experiment proved that Brücke had at least found pepsin to be present in the juice of flesh. This flesh was mostly muscle, but it must have consisted as well of fat, arteries, veins, nerves, etc.

Although Brücke thus obtained pepsin from a large piece of flesh, and references are made to the effect that in muscle as well as in most other tissues there is a diastatic enzyme of the nature of ptyalin or amylopsin, no one, so far as I can ascertain, has methodically taken up each tissue separately and made a glycerine extract of it, to ascertain whether any particular enzyme, or enzymes, exist in the different tissues.

Description of Method Adopted in this Research.

In consideration of the fact that Von Wittich's method of making glycerine extracts of tissues dissolved in most cases, at least, the enzymes which were present in the tissues, I adopted his method with slight modification. My object was not to determine the amount of the enzyme in the tissue, but to see if it were actually present. Otherwise, the task would have been an exceedingly difficult and laborious one, as various methods of extraction would have necessarily had to be followed in order to obtain the enzyme in its purest form, when it might be expected to show its greatest activity.

All tissues were subjected to the same process. They were all fresh, except in the case of those obtained from the post-mortem room. The tissues (normal and pathological) were macerated and put in alcohol before any putrefaction or other change could occur. The only tissues in which putrefaction might have occurred were post-mortem tissues. The greatest care was taken in thoroughly cleaning the vessels into which the tissues were placed, so as to get rid of extraneous germs. The tissues were minced in a mincing machine, and afterwards pounded in a mortar with powdered glass, until they were in a fine state of division. They were immersed in absolute alcohol for twenty-four hours. The alcohol was then allowed to evaporate at

the ordinary temperature of the room, the evaporation occurring in a large bell jar, in order to prevent dust falling into the vessel. The tissues were frequently powdered a second time when dry, and they were covered over with strong glycerine, the quantity of glycerine being much in excess of the bulk of the tissue. The vessel was then covered with a glass lid, and the extraction allowed to go on for a period of six to eight weeks. At the expiry of that time the contents were filtered through fine muslin, pressure being exerted to squeeze out any of the juice that remained in the tissue, and occasionally a little more glycerine was added to increase the quantity of the fluid. The fluid so obtained was a perfectly clear homogeneous fluid, and was now ready for experimental purposes. This method, as has been shown by Von Wittich, is a satisfactory one for demonstration purposes, but is by no means reliable for research, as the solutions contain enzymes in a far from pure state. Still, we know that most enzymes are soluble in glycerine, and, moreover, whether we are dealing with the pure enzyme or not, glycerine does extract it in sufficient quantity to give at all events qualitative results when used in digestion experiments.

The experiments were carried on in an incubator with heat regulator, so that any required temperature could be maintained.

The material consisted of fresh fibrin, starch solution, solution of cane sugar, solution of 0·2 per cent. HCl, solution of 1 per cent. Na_2CO_3, and the usual chemicals employed as tests in such researches. The starch solution was freshly prepared for each set of observations. It consisted of 1 grm. of the best rice starch dissolved in 50 c.c. of water. The cane sugar solution contained 1 grm. of sugar in 50 c.c. of water. The fibrin was fresh, and washed in running water for at least twelve hours before use.

If X be the name of the extract used, then X was divided into the following portions, and submitted to certain tests:—

1. 10 c.c. of X were added to 20 c.c. of starch solution. These two fluids were shaken in a test tube. The test tube was plugged, and placed in the incubator at a temperature of 38° C., for twenty-four hours.

The mixture was then tested with Fehling's solution,

and any reduction was noted. If there was any reduction, then the probability was that sugar had been formed, and the fluid was submitted to further tests. To 5 c.c. of the mixture were added 1 decigramme of phenyl-hydrazine hydrochloride, and 2 decigrammes of sodium acetate. The mixture was heated for half-an-hour, and the deposit which formed on cooling was examined microscopically for crystals of phenyl-glucosazone and phenyl-maltosazone.*

2. 10 c.c. of X were added to 1 grm. of fresh fibrin in beaker. The extract was diluted up to 40 c.c. of cold water. The beaker was covered with a glass lid, and placed in the incubator for twenty-four hours, at the same temperature (38° C.).

   The appearance of the fibrin was noted, and to a portion of the filtered fluid was added an equal quantity of sulphate of ammonium, and the presence or absence of a precipitate was observed.

3. 10 c.c. of X, diluted up to 40 c.c. with a 0·2 per cent. solution of hydrochloric acid, were added to 1 grm. of fibrin in beaker. The beaker was covered and placed in incubator as before.

   The appearance of the fibrin was noted, particular attention being paid to see whether there was any appearance of digestion. The biuret test was applied to the filtered solution, and the presence or absence of a rose pink hue observed.

4. 10 c.c. of X, diluted up to 40 c.c. with a 1 per cent. solution of carbonate of soda, were added to 1 grm. of fibrin in beaker. The beaker was covered and placed in incubator as before.

   The appearance of the fibrin was noted to see whether any erosion of it had occurred. A portion of the filtered fluid was examined by the biuret reaction, while another portion was evaporated down to a few

* I may state here that on no occasion did I observe the typical crystals which occur in sheaths and bundles. I obtained frequently crystals, yellow in colour, small, and almost amorphous in character.

drops, and examined microscopically for crystals of leucin or tyrosin.

On several occasions, when leucin or tyrosin were suspected, a portion of the filtered fluid was tested with Millon's reagent. The precipitate which formed was filtered off, and the filtrate evaporated down to small bulk. Any change in the colour of the solution was observed, and a few drops of the concentrated liquid were examined microscopically.

5. 10 c.c. of X were added to 20 c.c. of a solution of cane sugar. The two fluids were shaken in a test tube. The test tube was plugged, and placed, as before, in the incubator.

   The mixture was tested with Fehling, and any reduction noted. As in the case of 1, the phenyl-hydrazine test was frequently applied.

6. 10 c.c. of X were added to 50 c.c. of fresh milk diluted to 100 c.c. with water. The mixture was stirred, covered, and placed in incubator. Any special curdling of the milk was noted.

7. 10 c.c. of X were placed in a test tube, and put in incubator. The extract was then tested with Fehling's solution, and any reduction was noted.

In order to compare the results of the action on fibrin by the extract in alkaline and acid media, with the results in alkaline and acid media alone, confirmatory tests were frequently applied (the strengths of the solutions of hydrochloric acid and carbonate of soda being the same).

By means of these tests one was able to note:—

1. The conversion of starch by X into a reducing agent, and this probably by an enzyme similar in its action to ptyalin or amylopsin.
2. The change in fibrin when acted on by X in a watery solution, and the presence or absence of proteoses.
3. The change in fibrin when acted on by X in a 0·2 per cent. hydrochloric acid solution, and the presence or absence of peptones, the result of the activity of an enzyme similar to pepsin.

4. The change in fibrin when acted on by X in a 1 per cent. carbonate of soda solution, and the presence or absence of peptones, leucin, or tyrosin, the result of the activity of an enzyme similar to trypsin.
5. The inversion of cane sugar by X into a reducing sugar, and this probably by an enzyme similar in its action to inversin.
6. The curdling of milk, and this by an enzyme similar in its action to rennin.
7. Whether the extract itself had any reducing properties.

Sources of Error in the Experiments, and how these were avoided.

1. Length of time for extraction by glycerine—

We know that enzymes, when present in small amount (and they are likely to be so in the tissues), require considerable time for their extraction by glycerine. Consequently, little or no reaction might be obtained from tissues, although an enzyme was present, if the tissue were not a sufficiently long time in glycerine. To avoid this source of error, in all cases the tissues were immersed in glycerine for six weeks, and in many cases for a longer period.

2. Length of time required for enzymic action—

It is of importance to subject the solutions containing the supposed enzyme to a temperature of 38° C. for a considerable time. While enzymes may exist in the glycerine extract no reaction may be obtained, owing to a deficient exposure at the proper temperature of the mixed fluid under observation. To avoid this cause of error, I allowed the action to go on for a period of from eighteen to twenty-four hours.

3. The purity of the solutions used—

The solutions of starch and cane sugar must be fresh, and possess no reducing properties. Consequently, they must be always tested before any observation is made; and, further, these solutions must be tested after remaining in the incubator for twenty-four hours. I have found, in regard to this latter point, that a pure starch or cane sugar solution, when submitted to a temperature not exceeding 40° C. for twenty-four hours (with

the vessel or test tube in which the solution is contained plugged), should possess no reducing properties at the end of that time. Fehling's solution must be pure, and not alter in colour on boiling. With these precautions we are able to say, definitely, if the starch solution, plus X, reduces Fehling's fluid, that the extract itself has reducing properties, or that the starch has been converted into a substance that reduces Fehling's fluid. The first point is settled by testing the extract itself. If this has no reducing property, we may conclude that the starch solution has been altered by a substance which is present in X, which can reduce Fehling's fluid.

4. The presence of organisms in the tissues—

This question presents itself as we are aware that organisms and their ferments are capable of creating changes in starchy and proteid foods in a closely similar way to those caused by the unorganised ferments or enzymes that exist in the tissues. There are many chemical tests by which we may distinguish between the two classes of ferments, such as the use of peroxide of hydrogen, borax, salicylic acid (0·1 per cent.), thymol (0·5 per cent.), carbolic acid (0·5 to 1 per cent.), chloroform, and others, yet we are compelled to admit possible results depending on the existence of an organised which may be confused with those due to an unorganised ferment. It is necessary to make sure that no organisms enter during the preparation of the tissues. There must be no putrefactive change in the tissues under investigation, or in the fibrin itself. All beakers and test tubes must be sterilised, and before submitting their contents to the action of heat they must be sealed and plugged.

The organisms themselves are killed during the process of extraction and immersion in alcohol, but we have not to consider only the organisms, as they may be capable of liberating ferments or enzymes, which will be taken up by a suitable extractive. By the use of antiseptics we avoid this difficulty.

Although I never used antiseptics (as I intended to observe the results on the tissues unaltered), I hope in a future research to compare the results I have obtained with those in which antiseptics such as thymol or salicylic acid will be used.

The only tissues where such a difficulty really arose were those of the intestines of the rabbit and child, certain of the pathological tissues, in sputum and in the post-mortem tissues. In sputum, no doubt, pyogenic organisms exist in great numbers. The post-mortem tissues were removed in less than twenty-four hours after death, and were at once placed in absolute alcohol. The other tissues were fresh, and were removed, powdered, and placed in absolute alcohol within a few hours after their removal.

In the intestines putrefactive bacteria are always present, but the greatest care was taken in stripping off the mucous membrane of the bowel, and in washing it freely in running water before mincing and placing it in alcohol. The fibrin which was used was fresh, and contained no putrefactive organisms.

I admit that no means in the way of antiseptics have been used to distinguish whether the results depended on the action of the unorganised or organised ferments; but I consider that in most cases the results have not depended upon the organised, but upon the unorganised ferments or enzymes, which play such an important part in the process of digestion.

5. The cleavage of proteids by acids alone—

Fibrin is unaltered by the action of pepsin alone, but in the presence of hydrochloric acid rapid digestion takes place. A weak solution of the acid itself has the power of causing the fibrin to swell up and become translucent, and to produce an acid albumin, or even albumoses and peptones.

Do we know, then, whether the peptones that are produced in various experiments depend upon the activity of an enzyme in conjunction with HCl, or from HCl itself?

The biuret reaction is a fairly distinctive test.

If pepsin has been at work, then a rose-pink coloration results, but, if not, a violet coloration is produced.

6. The coagulation of milk—

A certain amount of coagulation occurs from heat alone, but the coagulation which thus occurs is very different from the form of clot produced by the action of rennin.

GLYCERINE EXTRACTS WERE MADE OF THE FOLLOWING TISSUES:—

I. Tissues from the rabbit—

(*a*) Bones; (*b*) small intestine; (*c*) large intestine; (*d*) blood; (*e*) stomach; (*f*) lungs; (*g*) kidneys; (*h*) liver; (*i*) muscle; (*j*) pancreas; (*k*) brain; (*l*) suprarenal bodies; (*m*) spleen; (*n*) heart; (*o*) hair and skin; (*p*) eyes.

II. Tissues from a human being (child)—

(*a*) Spinal cord; (*b*) heart; (*c*) muscle; (*d*) bone (partly ossified); (*e*) liver; (*f*) thyroid; (*g*) large intestine; (*h*) skin; (*i*) stomach; (*j*) vermiform appendix; (*k*) lung; (*l*) spleen; (*m*) suprarenal bodies; (*n*) brain; (*o*) kidneys; (*p*) small intestine; (*q*) gall bladder; (*r*) thymus gland; (*s*) pancreas; (*t*) cartilage; (*u*) fat.

III. Tissues from a human adult * (not post-mortem)—

(*a*) Tendo Achilles; (*b*) fat; (*c*) muscle; (*d*) cartilage; (*e*) ligament and synovial membrane; (*f*) bone; (*g*) skin; (*h*) connective tissue; (*i*) nerve; (*j*) placenta.

IV. Human post-mortem tissues (macroscopically and microscopically normal)—

(*a*) Liver (No. I.); (*b*) liver (No. II.); (*c*) lung; (*d*) skin; (*e*) large intestine; (*f*) kidneys; (*g*) spleen; (*h*) muscle; (*i*) small intestine; (*j*) fat.

V. Glycerine extracts were made of the following pathological tissues:—

(*a*) Carcinoma of skin (infected by cancer of pylorus); (*b*) scirrhus of breast; (*c*) sarcoma of face; (*d*) angeio-sarcoma of leg; (*e*) eclamptic tissues:—

(1) blood; (2) liver; (3) pancreas; (4) spleen; (5) brain; (6) kidneys;

(*f*) varicose veins; (*g*) tubercular sputum.

In my original paper, tables were submitted which showed the tests applied and the results obtained, but here it will be sufficient to deal with the results in a general manner, in short, to show whether these extracts have actions similar to those of ptyalin, pepsin, trypsin, inversin, and rennin.

* The first nine tissues were obtained from a healthy leg, removed by operation for sarcoma of the upper end of the femur.

(*a*) What is their Action in the Conversion of Starch into Sugar?

I have drawn out the following table to show a comparison of results obtained in the conversion of starch into sugar by the extracts of normal tissues. I have used the terms "abundant," "considerable," "distinct," etc., to denote relatively the density of the precipitate formed by their action in the reduction of Fehling, so as to give a clue to the amount of sugar formed, thus indicating, roughly, the activity of enzyme in the tissue extract or presumably its amount.

	I. Rabbit's Tissues.	II. Child's Tissues.	III. Human Adult Tissues.	IV. P.M. Tissues.
Abundant Conversion	Small Intestine X	...	...	Small Intestine X
	Large Intestine X	...	...	Large Intestine
	Stomach	...	...	...
	Liver X	Liver X	...	Liver (No. 2) X
	Muscle	...	Muscle	Muscle X
	Pancreas	Pancreas	Placenta	Lung
	Spleen	...	...	Spleen
Considerable Conversion	...	Lungs	...	Liver (No. 1) X
	Suprarenal bodies	Suprarenal bodies	...	...
	...	Kidneys	...	Kidneys
	...	Small Intestine	...	...
Distinct Conversion	...	Heart Muscle	...	...
	...	Large Intestine	...	...
	...	Brain	Fat X	...
	...	Thymus Gland	...	...
Slight Conversion	...	Muscle	Bone X	...
	Kidneys	Stomach	Skin X	...
	...	Vermiform Appendix	...	...
	Heart Muscle	Spleen	Connective Tissue	...
	...	Cartilage	Tendon X	...
	...	Fat X	...	Fat X
No Conversion	Bones	Bones	Cartilage	...
	Blood	Thyroid	Ligament	...
	Lungs	...	...	...
	Brain	Spinal Cord	Nerve	...
	Hair and Skin	Skin	...	Skin
	Eyes	Gall Bladder	...	...

A glance at this table shows that most of the succulent organs and tissues yield an extract which rapidly converts starch into sugar; while the drier tissues, such as bone, cartilage, etc., yield extracts which have no such power. In comparing the various tissues obtained from the rabbit, child, and adult, there is, on the whole, a similarity in their action.

Many of the tissues have an X marked opposite. These tissue extracts had the power of reducing Fehling themselves. Many tissues containing glycogen yield a sugar after their death. This fact may account for these extracts reducing Fehling, but in most cases the reduction of Fehling by the extract itself was slight as compared with the reduction by the starch solution previously acted on by the extract.

The following table shows the comparison of results obtained in the conversion of starch into sugar by the pathological tissues:—

Abundant Conversion,	Blood Liver Pancreas Spleen Brain Kidneys	Eclamptic Tissues.
Considerable Conversion,		
Distinct Conversion,	Carcinoma of Skin. Scirrhus of Breast. Angeio-sarcoma of Leg.	
Slight Conversion,	Sarcoma of Face. Varicose Veins. Tubercular Sputum.	
No Reaction,		

All the pathological extracts have the power of converting starch into sugar. Cancers and sarcomas do this markedly, while the various extracts of tissues that were examined from the patient who suffered and died from eclampsia have a very powerful action in this respect. One cannot say definitely that cancerous and sarcomatous tumours yield extracts which invariably convert starch into sugar, as a sufficient number have not been examined. The probability is, however, that this is so; and, moreover, soft or medullary carcinomata, and soft, round, or giant-celled sarcomata will probably have a greater power in causing this conversion than the hard scirrhus cancer or spindle-celled sarcoma.

Why should the tissues in eclampsia yield extracts which have such a powerful action in the conversion of starch? It is not that they contain more glycogen, as the extract itself would have in

that case reduced Fehling. Again, it is not probable that putrefactive organisms have had to do with this result, as in that case one would have expected something akin to tryptic digestion, which was always absent. Is there a special organism in this disease which has such a power, or, do the results depend upon the liberation of enzymes from the tissues in a greater abundance than exist normally? The tubercular sputum has a faint reaction in the conversion of starch. This result probably depends upon an organised ferment that is liberated after death from the pyogenic organisms which are present in such a sputum, or it may depend upon ptyalin in the saliva.

(*b*) What is their Action on Fibrin in a Watery Solution?

All tissues (normal and pathological) behave alike in yielding extracts which, with water alone, cause no change in fibrin; and when the solution is filtered and tested with sulphate of ammonium there is no precipitate which shows the presence of proteoses.

(*c*) What is their Action on Fibrin in an Acid Solution?

All the normal and pathological tissues have the power of more or less dissolving fibrin in a 0·2 per cent. HCl solution, and of yielding a solution of peptones which give the biuret reaction.*

The following tissue extracts have the power of dissolving fibrin more markedly than the others:—

Rabbit.	Child.	Adult.	Post-mortem.	Pathological.
Small Intestine	Large Intestine	...	...	...
Stomach	Stomach	...	...	...
Lungs	Lungs	...	Lung	...
Liver	Liver	...	Liver	...
Muscle	Muscle	Muscle	Muscle	...
...	...	...	...	Kidney } Eclamptic
...	...	...	...	Pancreas } Eclamptic

* This result cannot be due to the conversion of proteids into albuminoses, etc., by the acid itself, as fibrin subjected to the action of 40 c.c. of 0·2 per cent. HCl alone causes it to swell up, but not to be dissolved.

It will be seen that those tissues which have the greatest power in digesting fibrin correspond pretty closely in the different groups, and moreover correspond in great part to those tissues which yielded an extract that caused abundant conversion of starch into sugar.

(*d*) WHAT IS THEIR ACTION ON FIBRIN IN AN ALKALINE SOLUTION?

The only cases in which this occurred were :—

Rabbit.	Child.	Adult.	Post-mortem.	Pathological.
Small Int.	Small Int.	...	...	...
Large Int.	Large Int.	...	Large Int.	...
Pancreas	Pancreas	...	...	Pancreas
...	Liver	...	...	...

These results open up two questions :—

(1) As the reactions are so uniformly present in the intestines, and in no other tissues except pancreas and liver, do the results depend on organisms with their liberated ferments, or on an enzyme that is present in the tissues of a nature similar to trypsin of the pancreatic juice?

(2) Is the proteolytic ferment of the pancreatic juice soluble in glycerine, provided that the results do not depend upon organisms?

I do not intend to discuss these questions here. Still, with the exception of the large intestine obtained post-mortem, in which tissue organisms are likely to be present, I do not see how the other results can depend on bacteria, as the tissues were in every instance cleansed in running water before extraction, and were absolutely fresh. The question might have been settled had antiseptics been used; consequently, I am unable to oppose the views of Kühne,* and his school, or to agree with those of Hufner, but I think it probable that even with the use of antiseptics the same results would have probably occurred.

With regard to the second question, there is not the slightest doubt that the glycerine which was used extracted a small

* *Lehrb. d. Physiol. chem.*, 1868, S. 120.

quantity of trypsin, as the extract of the pancreas dissolved fibrin with the formation of peptones and crystals of leucin and tyrosin. Glycerine, however, extracts trypsin in small amount, and the solution obtained when placed with fibrin produces only a small quantity of peptones, and rarely crystals of leucin and tyrosin. To obtain a strong solution of trypsin one would have to adopt another method for its extraction, or to use a very watery solution of glycerine.

I do not think that it is at all likely that trypsin exists in many of the tissues, and the probability is that the proteolytic enzyme of the tissues is one which is similar in nature to pepsin.

(*e*) What is their Action in the Inversion of Cane Sugar into Dextrose?

The only tissues where there appeared to be inversion were :—

Rabbit.	Child.	Adult.	Post-mortem.	Pathological.
Liver X	Liver X	...	Liver No. I. X	...
...	...	...	Liver No. II. X	...
Pancreas	...	...	Lung X	...
...	Fat X	Fat X	Fat X	...
...	...	Bone X	L. Intestine X	...
...	...	Connective Tissue X	Muscle X	...
...	...	Tendon X	S. Intestine X	...
...	...	...	...	Sarcoma of face
...	...	...	...	Tubercular sputum

In most cases, then, the extract itself reduced Fehling and in almost all cases the reduction of Fehling depended on the extract, and not on an inversive ferment.

The extract of the pancreas of the rabbit, however, undoubtedly caused inversion of the cane sugar. This is strange, as the pancreas is not supposed to contain an invertive ferment. In no case did I obtain reactions proving the presence of *inversin* in the intestines.

The tubercular sputum rapidly inverted cane sugar, an action due probably to an organised ferment. The result is similar to that obtained from yeast. When the yeast is killed an organised ferment is liberated, which may be extracted by glycerine, and which inverts cane sugar into dextrose, as in the present instance.

(*f*) WHAT IS THEIR ACTION IN THE COAGULATION of MILK?

The only extracts which caused the coagulation of milk were :—

Rabbit.	Child.	Adult.	Post-mortem.	Pathological.
Stomach	Stomach	Placenta	...	...
Liver	...	...	Livers I. and II.	...
Pancreas	Pancreas	...	Lung	Pancreas (Eclamptic)
...	Large Int.	...	...	...

I shall now enumerate some of the more interesting results :—

I. EXTRACTS OF THE INTESTINES.

Paschutin * has proved that inversin can be obtained more effectively from the mucous membrane of the intestine than from the juice itself.

I, in no instance, obtained a reaction showing the presence of inversin in the intestines of the rabbit or child.

Is it possible that such an enzyme is not present in rabbit or child's intestines; or again, is it possible that glycerine failed to extract the enzyme inversin? Again, all the intestines examined yielded extracts which with 0·2 per cent. HCl had a marked action on fibrin. The same extracts had no action, or only doubtful action, in alkaline solutions. If we lay aside the action of organisms, which, if they had been present, would have caused digestion of fibrin in alkaline solutions, we have to conclude that the digestion is due to a ferment of the nature of pepsin which acts in an acid medium.

Is it not probable, then, that a proteolytic ferment is secreted by the intestinal mucous membrane which is related closely to pepsin?

Of course, in physiological conditions, pepsin would not exert its influence in the process of digestion, as the intestinal juice is alkaline.

We know that a juice is secreted from the upper part of the

* *Archiv. f. Anat. v. Physiol.*, 1871, pp. 305–384.

duodenum which contains pepsin. I think it probable that such an enzyme may exist along the whole intestinal mucous tract.

Again, it was easy to obtain, by glycerine extraction, the enzyme corresponding to ptyalin or amylopsin. In all cases this enzyme was extremely active. It is possible that in the child there is more use for this enzyme than in adults, as ptyalin of the saliva and amylopsin of the pancreatic juice may not be present in sufficient abundance at such an early age, while in the case of the rabbit there is a greater necessity for such a ferment, as the diet contains so much starch.

II. Extracts of the Stomach.

Glycerine extracts of the stomach of both rabbit and child not only gave reactions showing presence of pepsin and rennin, but also ptyalin or amylopsin. The conversion of starch into sugar in both these cases was very marked. I have not noticed in any text-book mention of ptyalin having been obtained by extraction of the mucous membrane of the stomach. This also may be a peculiarity of the stomach of the rabbit and that of the child, but it will be important in future to see what effect a glycerine extract of a well-washed mucous membrane of an adult stomach has upon starch.

III. Extract of Rabbit's Lung.

A very interesting result was obtained from the extract of the rabbit's lung. With 40 c.c. of 0·2 per cent. HCl, the extract caused 1 grm. of fibrin to become totally dissolved in a short time. The same result, although in a manner less marked, was obtained from the extract of the child's lung.

It seems strange that the lung of the rabbit should possess this power so markedly. Fibrin was digested by the extract of the lung as completely as by the extract of the stomach. Does the lung then contain pepsin in almost as active a form as it exists in the stomach? The significance of this result is not apparent.

IV. Extracts of the Pancreas.

Glycerine extracts of the pancreas of rabbit and child, and also of the pancreas from the eclamptic case, gave reactions which

showed the presence of pepsin in considerable amount. The fibrin was always totally dissolved in the acid solution. Does the pancreas then also contain pepsin? The pancreatic juice destroys the action of pepsin, as it is alkaline; and, consequently, even though pepsin be present, it has no influence on the digestion of food stuffs in the intestines. Still, it may be present in the pancreas all the same, and only exert its influence in certain forms of disease, or possibly when the intestinal juice becomes acid.

V. Extracts of the Liver.

Glycerine extracts of the liver invariably reduce Fehling, probably from the conversion of glycogen into a reducing sugar. In all cases, however, the reduction obtained by the starch solution, previously acted upon by the extract, was greater than that from the extract itself. It would appear that in the liver there is present an enzyme that corresponds to ptyalin.

There is also present an enzyme that corresponds to pepsin. In two instances, viz., liver of rabbit and human liver post-mortem, there was curdling of milk, produced by the action of the extract. In no cases was there a reaction suggesting tryptic activity.

VI. Extracts of Blood.

A glycerine extract of the blood of the rabbit, physiologically normal, had no reaction in the conversion of starch into sugar. On the other hand, the extract obtained from the blood of the eclamptic very rapidly converted starch into sugar.

VII. Extracts of Eclamptic Tissues.

All the extracts obtained from the tissues of the eclamptic had the power of converting starch into sugar very markedly, and also of partially digesting fibrin, while, with the exception of the pancreas, they had no action in alkaline solutions. These reactions must depend upon an altered condition of tissues in this disease, producing a greater quantity of active enzymes.

VIII. Extract of Tubercular Sputum.

It is interesting to note that a glycerine extract of tubercular

sputum has a marked inversive action. It has a faint power in the conversion of starch into sugar (probably from ptyalin in saliva), and in the digestion of fibrin in an acid medium. Both these reactions are slight, as compared with the inversive power. As I have mentioned before, the result is probably due to the liberation of an organised ferment from the pyogenic or other organisms which exist in sputum after their death.

IX. Extracts of Malignant Tumours.

The few carcinomata and sarcomata that were examined yielded extracts which converted starch into sugar, and also which digested fibrin slightly in an acid medium.

Do the above Results Depend then on the Activity of Enzymes?

This problem naturally presents itself, but I fail to see how any other explanation would account for the results. The glycerine extract itself has no reaction on starch or fibrin unless heated to the proper temperature, and kept at this temperature for a sufficient length of time. The extract must be treated in exactly the same manner as a solution containing a pure enzyme. In all respects there is proof that when a reaction occurred it depended upon enzymic activity. When starch was converted into sugar, this depended upon the enzyme ptyalin or amylopsin, or a similar enzyme. When fibrin was dissolved, and peptones were formed in an acid solution, then the enzyme pepsin was at work: or, again, when fibrin was dissolved, and peptones were formed in an alkaline solution, then trypsin was in action. When cane sugar was inverted into dextrose, this depended upon the enzyme inversin or a similar enzyme; and lastly, when the milk curdled, an enzyme similar in its action with rennin was at work.*

To go a step farther, it is probable that enzymes do not exist in the tissues as such, but in their parent zymogens, the enzymes being set free by a suitable extractive and in suitable media.

* In my original paper, I have discussed the questions more fully.

SUMMARY.

In the foregoing paper I have mentioned how the experiments were performed, and how certain difficulties which might lead to fallacies could be prevented. I traced the connection between enzymic activity of plant and animal life, showing that probably in the animal as well as in the plant an interstitial digestion was constantly at work. Although our knowledge of this question is still doubtful and obscure, one hopes that with the advancement of chemico-physiological science such a result may be confirmed, and may throw fresh light on the pathology of many obscure diseases. I then described the results of experiments on upwards of sixty extracts obtained by the glycerine process from the tissues of the rabbit, child, and the adult, both before and after death. Tables were next given of extracts of organs obtained in disease, and of tumours (sarcomata and carcinomata) and tubercular sputum. My results showed:—

(1) The presence of pepsin, or a substance analogous to it, in all the tissues, normal and pathological.

(2) The presence of a diastatic ferment in the larger proportion of the tissues examined—probably of the nature of ptyalin.

(3) The absence of trypsin in the tissues, except in the pancreas. Reactions which may have depended upon trypsin occurred in the intestines and in certain of the organs obtained post-mortem.

(4) That tissues which normally contained much glycogen formed an extract which reduced Fehling.

(5) That pepsin is present to a marked extent in the lung and liver of the rabbit as well as in the stomach.

(6) That the intestines contained a proteolytic ferment of the nature of pepsin. This result differs from that of most authorities.

(7) That an inversive ferment was not obtained by the glycerine process of extraction from the intestines of the child or rabbit.

(8) That an inversive ferment was rarely present in the tissues.

It was distinctly present in the extract of tubercular sputum.

(9) That a milk ferment, apart from those tissues in which it is known to exist, was rarely present.

(10) That the cancerous and sarcomatous tissues which were examined had proteolytic and distinctly diastatic properties.

(11) That rabbit's blood contained no diastatic enzyme, whereas eclamptic blood did.

(12) That all the tissues from the case of eclampsia yielded extracts which had marked diastatic properties, although these themselves did not reduce Fehling.

In conclusion, I may add that only a limited number of tissues have been examined, and that, before any final conclusions can be made as to the wide distribution of pepsin and ptyalin or amylopsin in physiological and pathological tissues, it would be necessary to examine similar tissues of many animals of the same and different species to see if all behave alike.

I cannot but entertain the hope that the examination of the blood in obscure diseases, and of carcinomatous and sarcomatous growths (with a view of ascertaining the presence or absence of enzymes), may throw light on the pathology and ætiology of certain diseases and morbid growths.

On the Law of Elastic Fatigue. By Dr W. Peddie.

(Read February 5, 1900.)

(Abstract.)

In this paper a discussion of the mode of description of the linear paths in the (log *b*, *n*) diagram—described in previous papers—was given. It was shown that the assumption that fatigue was induced by oscillation of a wire enabled one, in almost all cases, to predict accurately the mode of description of these paths. In a few cases, however, when the condition of the wire was such that the critical angle fell within the range of experimentally observed angles of oscillation, the mode of description of the linear paths agreed with the supposition that oscillation diminished fatigue. The truth of this supposition seemed also to be confirmed by the observed rate of decrease of oscillations in these cases; but further experimental evidence is required to fully test the point.

Observations on some Nemerteans from Singapore.

By **R. C. Punnett**, B.A. *Communicated by* Dr A. T. Masterman.

(Read May 7, 1900.)

The Nemerteans which form the subject of this communication were collected by Messrs F. P. Bedford and W. F. Lauchester during a year's stay in and near Singapore. The number of species procured is ten, nine of which have not hitherto been described. These are *Eupolia pholidota*, *Cerebratulus natans*, *C. brunneus*, *C. robustus*, *C. insignis*, *C. erythrus*, *C. sordidus*, *C. ulatiformius* and *C. bedfordii*. The species previously known is *Eupolia quinquelineata* (Bürger), but for reasons given it has been considered desirable to change the name to *E. melanogramma*. A careful examination of these forms has led to the following facts and conclusions of more general interest:—

(1) In one form (*E. melanogramma*) the excretory system possesses ducts which place its cavity in communication with that of the alimentary canal. The usual ducts to the exterior are also present. Consequently the alimentary canal in this region (*i.e.*, shortly behind the mouth) is placed in communication with the exterior by means of the excretory system and its ducts. Such a condition seems to find its closest parallel in the gill slits of the Chordata.

(2) In *Eupolia pholidota* the excretory ducts reach back into the intestinal region, thus co-existing in the same region as the series of gonidial ducts. Such a condition has not previously been noted in the group, and, taken in conjunction with the fact that the histology of the two ducts is different, seems to show that they are not serially homologous.

(3) In the genus *Eupolia*, the lateral nerve stems may either form a commissure above the anus, or else below, or even may terminate without forming a commissure at all. Such a fact tends to make one cautious in accepting the primitive nature, in such

forms as *Peripatus* and *Chiton*, of the supra-anal commissure upon which some writers have laid considerable stress.

(4) In the Lineidæ examined considerable variation occurs in the range and topography of the excretory system, as well as in the number of ducts. In none of the forms studied is to be found that incipient metamerism in the arrangement of the ducts which some observers claim to have demonstrated for other species.

(5) The vascular system of the Lineidæ shows but very little variation in the different species, except in so far as in the pre-cerebral region there may be either a well-marked head loop, or else a vascular network—a fact already pointed out by Bürger. It is worthy of note that there appears to be some correlation between the caudal extent of the excretory system and the point of exit of the dorsal blood-vessel from the proboscis sheath. This vessel in all the species here examined (with the exception of one case where the preservation was unsatisfactory for the determination of this point) leaves the sheath within a few micro-millimetres of the hinder termination of the excretory system, wherever that may be situated.

(6) The frontal organ characteristic of most Lineidæ is not always present.

(7) The structure of the skin is highly characteristic for each species.

The Theory of Alternants in the Historical Order of its Development up to 1841. By Thomas Muir, LL.D.

(Read March 19, 1900.)

The first traces of the special functions now known as *alternating functions* are said by Cauchy to be discernible in certain work of Vandermonde's; and if we view the functions as originating in the study of the number of values which a function can assume through permutation of its variables,* such an early date may in a certain sense be justifiable. To all intents and purposes, however, the theory is a creation of Cauchy's, and it is almost absolutely certain that its connection with determinants was never thought of until his time.

PRONY (1795).

[Leçons d'analyse. Considérations sur les principes de la méthode inverse des différences. *Journ. de l'Éc. Polyt.*, i. (pp. 211–273) pp. 264, 265.]

In the course of his investigations Prony comes upon a set of equations

$$\left.\begin{array}{l} \mu_1 + \mu_2 + \cdots + \mu_n = z_0 \\ \rho_1\mu_1 + \rho_2\mu_2 + \cdots + \rho_n\mu_n = z_1 \\ \rho_1^2\mu_1 + \rho_2^2\mu_2 + \cdots + \rho_n^2\mu_n = z_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \rho_1^{n-1}\mu_1 + \rho_2^{n-1}\mu_2 + \cdots + \rho_n^{n-1}\mu_n = z_{n-1} \end{array}\right\}$$

where the coefficients of each unknown are the 0th, 1st, 2nd, &c., powers of the same quantity, and where, therefore, the determinant of the set is that special form long afterwards known as the

* The history of this subject is referred to in Serret, M. J.-A.: "Sur le nombre de valeurs qui peut prendre une fonction quand on y permute les lettres qu'elle renferme," *Liouville's Journ. de Math.*, xv. pp. 1–70 (1849).

simplest form of alternant. The full solution is given for the first four cases, but without any indication of the method employed. Thus for four variables the results appear in the form

$$\mu_1 = \frac{-\rho_2\rho_3\rho_4 z_0 + (\rho_2\rho_3+\rho_2\rho_4+\rho_3\rho_4)z_1 - (\rho_2+\rho_3+\rho_4)z_2 + z_3}{(\rho_1-\rho_2)(\rho_1-\rho_3)(\rho_1 - \rho_4)},$$

$$\mu_2 = \frac{-\rho_1\rho_3\rho_4 z_0 + (\rho_1\rho_3+\rho_1\rho_4+\rho_3\rho_4)z_1 - (\rho_1+\rho_3+\rho_4)z_2 + z_3}{(\rho_2-\rho_1)(\rho_2-\rho_3)(\rho_2 - \rho_4)},$$

$$\mu_3 = \dots\dots$$

$$\mu_4 = \dots\dots$$

and the writer then adds:—

"En général, quelque soit le nombre n, pour avoir le numérateur de la fraction qui donne la constante μ_κ, il faut prendre toutes les racines, excepté la racine ρ_κ, et des $n-1$ racines restantes, en trouver le produit total, la somme des produits $n-2$ à $n-2$, $n-3$ à $n-3$, $n-4$ à $n-4$, ..., 2 à 2, 1 à 1, multiplier, respectivement, le produit total et chacune des sommes par $z_0, z_1, z_2, \dots, z_{n-2}$, ajouter z_{n-1}, et donner à tous les termes des signes alternatifs, en commençant par $-$ ou $+$, selon que n est pair ou impair.

"Pour avoir le dénominateur, on soustraira, successivement, de ρ_κ chacune des autres racines, et on fera un produit de toutes les différences données par ces soustractions."

It is, of course, quite possible that Prony was not acquainted with Vandermonde's memoir of 1771, or Laplace's of 1772, or Bezout's of 1779; and, further, that in seeking for the solution of his equations he was lucky enough to hit upon the set of multipliers which, being used, would, on the performance of addition, eliminate all the unknowns except one; *e.g.*, in the case of four variables the multipliers

$$\begin{aligned} &-\rho_2\rho_3\rho_4, \\ &+(\rho_2\rho_3+\rho_2\rho_4+\rho_3\rho_4), \\ &-(\rho_2+\rho_3+\rho_4), \\ &\quad 1. \end{aligned}$$

If, however, he was familiar with the method of any one of these memoirs, and applied it to the set of equations under discus-

sion, it would scarcely be possible for him not to anticipate Cauchy and Schweins in the discovery of the elementary properties of alternants. Thus, to take again the case of four variables, say the equations

$$\left.\begin{array}{l} x + y + z + w = p \\ ax + by + cz + dw = q \\ a^2x + b^2y + c^2z + d^2w = r \\ a^3x + b^3y + c^3z + d^3w = s \end{array}\right\},$$

Laplace's process would have given the value of x in the form

$$\frac{|b^1c^2d^3|p - |b^0c^2d^3|q + |b^0c^1d^3|\,r - |b^0c^1d^2|s}{|b^1c^2d^3| \;\; - |b^0c^2d^3|a + |b^0c^1d^3|a^2 - |b^0c^1d^2|a^3},$$

and Prony obtaining it in the form

$$\frac{bcd\,.\,p \; - (bc+bd+cd)q + (b+c+d)r \; - s}{bcd\,.\,a^0 - (bc+bd+cd)a + (b+c+d)a^2 - a^3}$$

could not have failed to know in their general forms the theorems

$$\begin{aligned} |b^1c^2d^3| \div |b^0c^1d^2| &= bcd\,, \\ |b^0c^2d^3| \div |b^0c^1d^2| &= bc+bd+cd\,, \\ |b^0c^1d^3| \div |b^0c^1d^2| &= b+c+d\,, \end{aligned}$$

and

$$|a^0b^1c^2d^3| \div |b^0c^1d^2| = (d-a)(c-a)(b-a)\,,$$

and $\therefore$ $$|a^0b^1c^2d^3| = (d-a)(c-a)(b-a)(c-b)(c-a)(b-a).$$

CAUCHY (1812).

[Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions operées entre les variables qu'elles renferment. *Journ. de l'Éc. Polyt.*, x. pp. 29–51, 51–112.]

By reason of the fact that Cauchy viewed determinants as a class of alternating functions, it has already been necessary to give an account * of a considerable portion of the first part (pp. 29–51) of this memoir: in fact, only five pages (pp. 45–51) remain to be dealt with if the portion referred to be borne in mind.

* See *Proc. Roy. Soc. Edinb.*, xiv. pp. 499–502.

From observing the substitutions which result in the vanishing of the function, he derives the following theorem :—

"Soit $S(\pm K)$ une fonction symétrique alternée quelconque. Désignons par α, β, γ, &c., les indices qu'elle renferme, et par

$$\begin{array}{llll} a_\alpha, & a_\beta, & a_\gamma, & \ldots . \\ b_\alpha, & b_\beta, & b_\gamma, & \ldots . \\ c_\alpha, & c_\beta, & c_\gamma, & \ldots . \\ \ldots & \ldots & \ldots & \ldots \end{array}$$

les quantités qui dans cette fonction se trouvent affectées des indices α, β, γ, ... Si l'on remplace

$$b_\alpha, c_\alpha, \ldots . \quad b_\beta, c_\beta, \ldots . . \quad b_\gamma, c_\gamma, \ldots . .$$

par des fonctions semblables des quantités a_α, a_β, a_γ, ... ; la fonction symétrique alternée deviendra divisible par chacune des quantités

$$\begin{array}{l} a_\alpha - a_\beta, \\ a_\alpha - a_\gamma, \\ \ldots\ldots \\ a_\beta - a_\gamma, \\ \ldots\ldots \text{"} \end{array}$$

From this he passes to alternating functions "which contain only one kind of quantities," and deduces the result that

$$S(\pm a_1^p a_2^q \ldots . a_n^t) \text{ is divisible by}$$
$$(a_2 - a_1)(a_3 - a_1) \ldots . . (a_n - a_1)(a_3 - a_2) \ldots . (a_n - a_2) \ldots . (a_n - a_{n-1}).$$

The question as to the remaining factor is then dealt with in the three simplest cases :—

(1) In the case of $S(\pm a_1^0 a_2^1 \ldots . a_n^{n-1})$ it is found as follows to be 1.

"La somme des exposans des lettres a_1, a_2, ... a_n dans chaque terme de la fonction symétrique alternée

$$S(\pm a_1^0 a_2^1 a_3^2 \ldots . a_{n-1}^{n-2} a_n^{n-1})$$

sera

$$0 + 1 + 2 + \ldots . + (n-2) + (n-1) = \frac{n(n-1)}{2}.$$

Mais les facteurs du produit A [*i.e.*, $(a_2 - a_1) \ldots . (a_n - a_{n-1})$]

étant aussi en nombre égal à $\frac{1}{2}n(n-1)$, la somme des exposans des lettres $a_1, a_2, \ldots, a_n$ dans chaque terme du développement de ce produit sera encore égale à ce nombre ; par suite, le quotient qu'on obtiendra, en divisant la fonction symétrique alternée par le produit, sera une quantité constante. Soit c la quantité dont il s'agit, on aura

$$S^n\left(\pm \overset{0}{a_1}\overset{1}{a_2}\overset{2}{a_3} \ldots \overset{n-1}{a_n}\right) = cA.$$

Pour déterminer c on observera que le terme

$$\overset{0}{a_1}\overset{1}{a_2}\overset{2}{a_3} \ldots \overset{n-1}{a_n}$$

a pour coefficient l'unité dans la fonction donnée et dans le produit A ; on doit donc avoir $c=1$."

Before proceeding to the next case he recalls the fact that

the product or quotient of two alternating functions of order n *is a symmetric function of the same order*,

and is thus enabled to amplify one of the preceding propositions by affirming that

the result of dividing $S\left(\pm \overset{p}{a_1}\overset{q}{a_2} \ldots \overset{t}{a_n}\right)$ *by* $S\left(\pm \overset{0}{a_1}\overset{1}{a_2} \ldots \overset{n-1}{a_n}\right)$ *is a symmetric function of* $a_1, a_2, \ldots, a_n$.

(2) In the case of $S\left(\pm \overset{0}{a_1}\overset{1}{a_2} \ldots \overset{n-2}{a_{n-1}}\overset{n}{a_n}\right)$ the quotient is found to be $a_1 + a_2 + \ldots + a_n$.

For the quotient "sera nécessairement du premier degré par rapport aux quantités $a_1, a_2, \ldots, a_n$: et comme elle doit être symétrique et permanente par rapport à ces quantités, on sera obligé de supposer égale à

$$c(a_1 + a_2 + \ldots + a_n) = cS^n(a_1),$$

c étant une constante qui ne peut différer ici de l'unité."

(3) In the case of $S\left(\pm \overset{1}{a_1}\overset{2}{a_2} \ldots \overset{n}{a_n}\right)$ the quotient is, of course, found to be $a_1a_2 \ldots a_n$.

The memoir closes with the conditions for the identity of two alternating functions, these being stated to be (1) that all the terms of the first function be contained in the second; (2) that the terms have the same numerical coefficients in both; (3) that one of the terms of the first has the same sign as the corresponding term of the second.

SCHWEINS (1825).

[THEORIE DER DIFFERENZEN UND DIFFERENTIALE; u. s. w. Von Ferd. Schweins. vi. + 666 pp. Heidelberg, 1825. Pp. 317–431: *Theorie der Producte mit Versetzungen.*]

It may be remembered that Schweins' large volume contains seven separate treatises, that the third treatise deals with determinants (*Producte mit Versetzungen*), and is divided into four sections (*Abtheilungen*). The first of the four almost entirely concerns *general* determinants, and consequently an account of it has already been given. The second section (pp. 369–398) now falls to be undertaken, its heading being "Determinants in which the upper index denotes a power" (*Producte mit Versetzungen, wenn die oberen Elemente das Potentiiren angeben*).

His first theorem is

$$\overset{h}{A_1}\overset{h}{A_2}\overset{h}{A_3}\ldots\overset{h}{A_n}\cdot\Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{A_3}\ldots\ldots\overset{a_n}{A_n}\Big) = \Big\|\overset{h+a_1}{A_1}\overset{h+a_2}{A_2}\overset{h+a_3}{A_3}\ldots\ldots\overset{h+a_n}{A_n}\Big)$$

which is seen to be an extension of one of Cauchy's; but, besides this, in the first chapter there is practically nothing worth noting.

The remaining four chapters, however, are full of interest, and deserve every attention, as until the present day they have been utterly lost sight of and contain a theorem or two which are still quite new.

The second chapter concerns the multiplication of an alternant of the n^{th} order by the sum of the p-ary combinations of the variables in their h^{th} power. In Schweins' notation this product is represented by

$$(\overset{h}{A_1}, \overset{h}{A_2}, \ldots, \overset{h}{A_n})^p \,.\, \Big\|\overset{a_1}{A_1}\overset{a_2}{A_2}\ldots\ldots\overset{a_n}{A_n}\Big);$$

in later notation, the case where $n=3$, $p=2$, $h=5$ would be written

$$(a^5b^5 + a^5c^5 + b^5c^5) \,.\, \begin{vmatrix} a^r & a^s & a^t \\ b^r & b^s & b^t \\ c^r & c^s & c^t \end{vmatrix}, \quad \text{or } \Sigma a^5b^5 \,.\, |a^rb^sc^t|.$$

The case where $p=1$ is first dealt with, and the proof is written

out at length without specialising n; but as this does not add to clearness or conviction, n may here, for convenience in writing, be taken $=4$. Let, then, the alternant be

$$|a^r b^s c^t d^u|$$

so that the multiplier is

$$a^h + b^h + c^h + d^h .$$

Expanding the multiplicand first according to powers of a, we perform the multiplication by a^h; expanding next according to powers of b, we perform the multiplication by b^h; and so on, the sum of the products being naturally arrangeable as a square array of sixteen terms, viz.,

$$\begin{array}{rcccc} & a^{r+h}|b^s c^t d^u| - & a^{s+h}|b^r c^t d^u| + & a^{t+h}|b^r c^s d^u| - & a^{u+h}|b^r c^s d^t| \\ - & b^{r+h}|a^s c^t d^u| + & b^{s+h}|a^r c^t d^u| - & b^{t+h}|a^r c^s d^u| + & b^{u+h}|a^r c^s d^t| \\ + & c^{r+h}|a^s b^t d^u| - & c^{s+h}|a^r b^t d^u| + & c^{t+h}|a^r b^s d^u| - & c^{u+h}|a^r b^s d^t| \\ - & d^{r+h}|a^s b^t c^u| + & d^{s+h}|a^r b^t c^u| - & d^{t+h}|a^r b^s c^u| + & d^{u+h}|a^r b^s c^t| . \end{array}$$

Recombination of these, however, is possible by taking them in vertical sets of four, and the result of doing this is

$$|a^{r+h} b^s c^t d^u| - |a^{s+h} b^r c^t d^u| + |a^{t+h} b^r c^s d^u| - |a^{u+h} b^r c^s d^t| ;$$

so that we have

$$|a^r b^s c^t d^u| \,.\, \Sigma a^h = |a^{r+h} b^s c^t d^u| + |a^r b^{s+h} c^t d^u| + |a^r b^s c^{t+h} d^u| + |a^r b^s c^t d^{u+h}| ,$$

and generally

$$\begin{aligned} |a^r b^s c^t d^u e^v \ldots\ldots| \,.\, \Sigma a^h = |a^{r+h} b^s c^t d^u e^v \ldots\ldots| &+ |a^r b^{s+h} c^t d^u e^v \ldots\ldots| \\ &+ |a^r b^s c^{t+h} d^u e^v \ldots\ldots| + \ldots\ldots \end{aligned}$$

The special case where $r, s, t, u, \ldots\ldots$ proceed by a common difference, h, is drawn attention to, as then all the alternants on the right vanish except the last: that is to say, we have

$$|a_1^r a_2^{r+h} a_3^{r+2h} \ldots\ldots a_n^{r+(n-1)h}| \,.\, \Sigma a_1^h = |a_1^r a_2^{r+h} a_3^{r+2h} \ldots\ldots a_{n-1}^{r+(n-2)h} a_n^{r+nh}| ,$$

a result which may be looked upon as an immediate generalisation of one of Cauchy's.

When $p > 1$, the mode of proof is totally different, being an attempt at so-called "mathematical induction." It is not by any means readily convincing, and is much less so than it might have

been, as, although there are *two* general integers involved, viz., p and n, Schweins attends only to the second of them. He begins with the case of $n=4$, $p=2$,—that is to say, the multiplication of

$$|a^r b^s c^t d^u| \text{ by } \Sigma a^h b^h ,$$

the result being

$$\begin{aligned}
\left(A_1^h, A_2^h, A_3^h, A_4^h\right)^{(2)} . \left\|A_1^{a_1} A_2^{a_2} A_3^{a_3} A_4^{a_4}\right) = \; & \left\|A_1^{h+a_1} A_2^{h+a_2} A_3^{a_3} A_4^{a_4}\right) \\
& + \left\|A_1^{h+a_1} A_2^{a_2} A_3^{h+a_3} A_4^{a_4}\right) \\
& + \left\|A_1^{h+a_1} A_2^{a_2} A_3^{a_3} A_4^{h+a_4}\right) \\
& + \left\|A_1^{a_1} A_2^{h+a_2} A_3^{h+a_3} A_4^{a_4}\right) \\
& + \left\|A_1^{a_1} A_2^{h+a_2} A_3^{a_3} A_4^{h+a_4}\right) \\
& + \left\|A_1^{a_1} A_2^{a_2} A_3^{h+a_3} A_4^{h+a_4}\right) .
\end{aligned}$$

To indicate the mode of formation of the alternants on the right from the given alternant on the left, he says :—

"Hier entstehen alle Vertheilungen von h, h zu zweien in vier Abtheilungen, nämlich

$h+a_1$	$h+a_2$	a_3	a_4
$h+a_1$	a_2	$h+a_3$	a_4
$h+a_1$	a_2	a_3	$h+a_4$
a_1	$h+a_2$	$h+a_3$	a_4
a_1	$h+a_2$	a_3	$h+a_4$
a_1	a_2	$h+a_3$	$h+a_4$

."

He next takes the case where $n=5$ and $p=3$: that is to say, the case of

$$|a^r b^s c^t d^u e^v| . \Sigma a^h b^h c^h ,$$

and gives as his result

$$\text{“}\left(A_1^h, A_2^h, A_3^h, A_4^h, A_5^h\right)^{(3)} . \left| A_1^{a_1} A_2^{a_2} A_3^{a_3} A_4^{a_4} A_5^{a_5} \right)$$

$$= \left\| A_1^{h+a_1} A_2^{h+a_2} A_3^{h+a_3} A_4^{a_4} A_5^{a_5} \right)$$

$$+ \left\| A_1^{h+a_1} A_2^{h+a_2} A_3^{a_3} A_4^{h+a_4} A_5^{a_5} \right)$$

$$+ \dots\dots\dots\dots\dots\dots$$

$$+ \left\| A_1^{a_1} A_2^{a_2} A_3^{h+a_3} A_4^{h+a_4} A_5^{h+a_5} \right) ,$$

wo h, h, h in fünf Abtheilungen zu dreien vertheilt werden, nämlich

$h+a_1$	$h+a_2$	$h+a_3$	a_4	a_5
$h+a_1$	$h+a_2$	a_3	$h+a_4$	a_5
$h+a_1$	$h+a_2$	a_3	a_4	$h+a_5$
$h+a_1$	a_2	$h+a_3$	$h+a_4$	a_5
$h+a_1$	a_2	$h+a_3$	a_4	$h+a_5$
$h+a_1$	a_2	a_3	$h+a_4$	$h+a_5$
a_1	$h+a_2$	$h+a_3$	$h+a_4$	a_5
a_1	$h+a_2$	$h+a_3$	a_4	$h+a_5$
a_1	$h+a_2$	a_3	$h+a_4$	$h+a_5$
a_1	a_2	$h+a_3$	$h+a_4$	$h+a_5$

,

the table being intended to make clear the fact that the five indices of each of the ten alternants on the right of the identity is got from the five

$$a_1, a_2, a_3, a_4, a_5$$

of the given alternant on the left by adding h to three of them.

The mode of formation, seen to hold in these two cases, being then supposed to hold for

$$\left(A_1^h, A_2^h, \ldots, A_{n-1}^h\right)^{(p)} \cdot \left|A_1^{a_1} A_2^{a_2} \ldots A_{n-1}^{a_{n-1}}\right),$$

is attempted to be shown to hold for

$$\left(A_1^h, A_2^h, \ldots, A_{n-1}^h, A_n^h\right)^{(p)} \cdot \left|A_1^{a_1} A_2^{a_2} \ldots A_{n-1}^{a_{n-1}} A_n^{a_n}\right);$$

that is to say, the case for n variables, $A_1, \ldots, A_n$ is sought to be made dependent on the case for $n-1$ variables, $A_1, \ldots, A_{n-1}$, p remaining the same in both. The process followed is to change the first factor into

$$\left(A_1^h, A_2^h, \ldots, A_{x-1}^h, A_{x+1}^h, \ldots, A_n^h\right)^{(p)}$$
$$+\left(A_1^h, A_2^h, \ldots, A_{x-1}^h, A_{x+1}^h, \ldots, A_n^h\right)^{(p-1)} \cdot A_x^h,$$

express the second factor—the alternant—in terms of n alternants of the $(n-1)^{\text{th}}$ order, and then perform the required multiplication and condense the result. This being satisfactorily accomplished, it would not of course follow from the two special cases previously dealt with that the theorem had been established in all its generality, but merely that it held for any number of variables $A_1, A_2, \ldots$ so long as p was not greater than 3. The passage from one value of p to the next higher—which is left unattempted by Schweins—is not free from difficulty, as will be seen on trying a particular instance,—say the passage from

$$|a^r b^s c^t d^u| \cdot (a^h b^h + a^h c^h + a^h d^h + b^h c^h + b^h d^h + c^h d^h)$$

to

$$|a^r b^s c^t d^u| \cdot (a^h b^h c^h + a^h b^h d^h + a^h c^h d^h + b^h c^h d^h).$$

Several special cases of the general theorem are noted, where a number of the alternants on the right vanish and where consequently a comparatively simple result is attained.

The first of these is where the indices of the alternant to be multiplied proceed by a common difference h: the identity then is

$$\left(A_1^h, A_2^h, \ldots, A_n^h\right)^{(p)} \cdot \left|A_1^{a+h} A_2^{a+2h} \ldots\ldots A_n^{a+nh}\right)$$
$$= \left|A_1^{a+h} A_2^{a+2h} \ldots\ldots A_{n-p}^{a+(n-p)h} A_{n-p+1}^{a+(n-p+2)h} \ldots\ldots A_n^{a+(n+1)h}\right).$$

The second is where $h = -h$, and the indices proceed by a common difference h, the result then being

$$\left(A_1^{-h}, A_2^{-h}, \ldots, A_n^{-h}\right)^{(p)} . \left| A_1^{a+h} A_2^{a+2h} \ \ldots\ldots\ A_n^{a+nh} \right)$$
$$= \left| A_1^{a}\ A_2^{a+h} \ \ldots\ldots\ A_p^{a+(p-1)h} A_{p+1}^{a+(p+1)h} \ \ldots\ldots\ A_n^{a+nh} \right).$$

The third is where the series of indices consists of two progressions proceeding by the common difference h, and where, of course, there are fewer vanishing terms in the product.

In the next chapter the subject matter is quite similar: in fact, the only difference is in the constitution of the multiplier, which is more extensive than before by reason of the fact that in forming the p-ary combinations there is now no restriction as to non-repetition of an element. Thus, instead of the example

$$|a^r b^s c^t| . (a^h b^h + a^h c^h + b^h c^h)$$

we should now have

$$|a^r b^s c^t| . (a^h b^h + a^h c^h + b^h c^h + a^{2h} + b^{2h} + c^{2h}).$$

The method followed is exactly the same as before. Three simple cases are carefully worked out, viz.,

$$|a^r b^s| . (a^{2h} + b^{2h} + a^h b^h),$$
$$|a^r b^s c^t| . (a^{2h} + b^{2h} + c^{2h} + a^h b^h + a^h c^h + b^h c^h),$$
$$|a^r b^s c^t| . (a^{3h} + b^{3h} + c^{3h} + a^{2h} b^h + a^{2h} c^h + b^{2h} a^h + b^{2h} c^h + c^{2h} a^h + c^{2h} b^h + a^h b^h c^h),$$

the results in Schweins' notation—where the change to rectangular brackets should be noted—being

$$\left[A_1^h, A_2^h\right]^{(2)} . \left|A_1^{a_1} A_2^{a_2}\right) = \left|A_1^{2h+a_1} A_2^{a_2}\right) + \left|A_1^{h+a_1} A_2^{h+a_2}\right) + \left|A_1^{a_1} A_2^{2h+a_2}\right),$$

$$\left[A_1^h, A_2^h, A_3^h\right]^{(2)} . \left|A_1^{a_1} A_2^{a_2} A_3^{a_3}\right) = \left|A_1^{2h+a_1} A_2^{a_2} A_3^{a_3}\right) + \left|A_1^{a_1} A_2^{2h+a_2} A_3^{a_3}\right)$$
$$+ \left|A_1^{a_1} A_2^{a_2} A_3^{2h+a_3}\right) + \left|A_1^{h+a_1} A_2^{h+a_2} A_3^{a_3}\right)$$
$$+ \left|A_1^{h+a_1} A_2^{a_2} A_3^{h+a_3}\right) + \left|A_1^{a_1} A_2^{h+a_2} A_3^{h+a_3}\right),$$

$$\left[A_1^h, A_2^h, A_3^h\right]^{(3)} . \left|A_1^{a_1} A_2^{a_2} A_3^{a_3}\right) = \left|A_1^{3h+a_1} A_2^{a_2} A_3^{a_3}\right) + \left|A_1^{a_1} A_2^{3h+a_2} A_3^{a_3}\right)$$
$$+ \left|A_1^{a_1} A_2^{a_2} A_3^{3h+a_3}\right) + \left|A_1^{2h+a_1} A_2^{h+a_2} A_3^{a_3}\right)$$
$$+ \left|A_1^{2h+a_1} A_2^{a_2} A_3^{h+a_3}\right) + \left|A_1^{a_1} A_2^{2h+a_2} A_3^{h+a_3}\right)$$
$$+ \left|A_1^{h+a_1} A_2^{2h+a_2} A_3^{a_3}\right) + \left|A_1^{h+a_1} A_2^{a_2} A_3^{2h+a_3}\right)$$
$$+ \left|A_1^{a_1} A_2^{h+a_2} A_3^{2h+a_3}\right) + \left|A_1^{h+a_1} A_2^{h+a_2} A_3^{h+a_3}\right).$$

Each result is seen, as in the preceding case, to be a sum of alternants differing only in the indices from the alternant which is the subject of multiplication. Further, it is observed that this difference is a difference in excess, the indices of the multiplicand appearing in all the terms of the product, so that the only difficulty is to ascertain what addendum is to be made to each. The next observation is that the addendum is a multiple of h, and that in the three cases the multiples are the following:—

$$\begin{array}{lll}
2h,\ 0h & 2h,\ 0h,\ 0h & 3h,\ 0h,\ 0h \\
1h,\ 1h & 0h,\ 2h,\ 0h & 0h,\ 3h,\ 0h \\
0h,\ 2h & 0h,\ 0h,\ 2h & 0h,\ 0h,\ 3h \\ \cline{2-3}
 & 1h,\ 1h,\ 0h & 2h,\ 1h,\ 0h \\
 & 1h,\ 0h,\ 1h & 2h,\ 0h,\ 1h \\
 & 0h,\ 1h,\ 1h & 0h,\ 2h,\ 1h \\ \cline{3-3}
 & & 1h,\ 2h,\ 0h \\
 & & 1h,\ 0h,\ 2h \\
 & & 0h,\ 1h,\ 2h \\ \cline{3-3}
 & & 1h,\ 1h,\ 1h\,.
\end{array}$$

The law of formation seen by Schweins in these coefficients of h is to be gathered from the sentence: "Hier werden alle mögliche Zerfällungen einer Zahl in mehrere Abtheilungen gebracht," and is nothing more nor less than the solution of the problem of putting p things in every possible way into n compartments. Thus, to take another example, if p were 2 and n were 4, the coefficients would be

$$\begin{array}{llll}
2, & 0, & 0, & 0 \\
0, & 2, & 0, & 0 \\
0, & 0, & 2, & 0 \\
0, & 0, & 0, & 2 \\
\\
1, & 1, & 0, & 0 \\
1, & 0, & 1, & 0 \\
1, & 0, & 0, & 1 \\
0, & 1, & 1, & 0 \\
0, & 1, & 0, & 1 \\
0, & 0, & 1, & 1\,.
\end{array}$$

Assuming this law to hold in the case of $n-1$ variables $A_1, \ldots, A_{n-1}$, his mode of writing it being

$$\left[A_1^h,\ A_2^h, \ldots, A_{n-1}^h\right]^{(p)} \cdot \left\|A_1^{a_1} A_2^{a_2} \ldots . A_{n-1}^{a_{n-1}}\right) = \sum\nolimits_{p,n-1} \left\|A_1^{ph+a_1} A_2^{a_2} \ldots A_{n-1}^{a_{n-1}}\right),$$

he tries to show that it will hold in the case of one additional variable A_n, the possible variation of p being ignored as before. To do this he changes the factor

$$\left[A_1^h, A_2^h, \ldots, A_n^h\right]^{(p)}$$

into

$$\left[A_1^h, A_2^h, \ldots, A_{n-1}^h\right]^{(p)} + \left[A_1^h, A_2^h, \ldots, A_{n-1}^h\right]^{(p-1)} . A_n^h ,$$

and the second factor exactly as it was changed in the preceding chapter, performs the required multiplication, and condenses the result.

The rest of the chapter is occupied with the consideration of special cases, the lines of specialisation being exactly those followed in the case of the previous general theorem. Only the first need be noted: it is

$$\begin{aligned}\left[A_1^h, A_2^h, \ldots, A_n^h\right]^{(p)} . &\left\|A_1^{a+h} A_2^{a+2h} \quad \ldots A_n^{a+nh}\right) \\ &= \left\|A_1^{a+h} A_2^{a+2h} \quad \ldots . A_{n-1}^{a+(n-1)h} A_n^{a+(n+p)h}\right).\end{aligned}$$

The fourth chapter does not impress one favourably, although the author speaks of its importance in connection with later investigations. It is almost entirely dependent on a very special case of the theorem of the second chapter, viz., the case where all the indices, except the last, of the multiplicand proceed by a common difference h, and where consequently all the alternants in the result vanish except two. In the original notation it is

$$\begin{aligned}\left(A_1^h, A_2^h, \ldots, A_n^h\right)^{(n-p)} . &\left\|A_1^{a+h} A_2^{a+2h} \quad \ldots . A_{n-1}^{a+(n-1)h} A_n^s\right) \\ = &\left\|A_1^{a+h} \ldots A_p^{a+ph} A_{p+1}^{a+(p+2)h} \quad \ldots . A_{n-1}^{a+nh} A_n^{s+h}\right) \\ + &\left\|A_1^{a+h} \ldots A_{p-1}^{a+(p-1)h} A_p^{a+(p+1)h} \quad \ldots . A_{n-1}^{a+nh} A_n^s\right),\end{aligned}$$

but for convenience in what follows it may be shortly written

$$N_{n-p} . A_s = M_{p+1,s+h} + M_{p,s} .$$

Using it $n-p+1$ times in succession we have

$$\begin{aligned}
N_{n-p} \,.\, A_s &= M_{p+1,s+h} + M_{p,s}, \\
-N_{n-p-1} \,.\, A_{s+h} &= -M_{p+2,s+2h} - M_{p+1,s+h} \\
N_{n-p-2} \,.\, A_{s+2h} &= M_{p+3,s+3h} + M_{p+2,s+2h} \\
-N_{n-p-3} \,.\, A_{s+3h} &= -M_{p+4,s+4h} - M_{p+3,s+3h} \\
&\cdots\cdots\cdots\cdots \\
(-)^{n-p}N_0 \,.\, A_{s+(n-p)h} &= 0 + (-)^{n-p}M_{n,s+(n-p)h}
\end{aligned}$$

and therefore by addition

$$M_{p,s} = N_{n-p} \,.\, A_s - N_{n-p-1} \,.\, A_{s+h} + N_{n-p-2} \,.\, A_{s+2h} - \ldots (-)^{n-p}N_0 \,.\, A_{s+(n-p)h},$$

or

$$\begin{aligned}
&\left|A_1^{a+h} A_2^{a+2h} \ldots A_{p-1}^{a+(p-1)h} A_p^{a+(p+1)h} \ldots\ldots A_{n-1}^{a+nh} A_n^{s}\right) \\
&= \left(A_1^h, A_2^h, \ldots\ldots A_n^h\right)^{(n-p)} \,.\, \left|A_1^{a+h} \ldots\ldots A_{n-1}^{a+(n-1)h} A_n^{s}\right) \\
&\quad - \left(A_1^h, A_2^h, \ldots\ldots A_n^h\right)^{(n-p-1)} \,.\, \left|A_1^{a+h} \ldots\ldots A_{n-1}^{a+(n-1)h} A_n^{s+h}\right) \\
&\quad + \left(A_1^h, A_2^h, \ldots\ldots A_n^h\right)^{(n-p-2)} \,.\, \left|A_1^{a+h} \ldots\ldots A_{n-1}^{a+(n-1)h} A_n^{s+2h}\right) \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad + (-1)^{n-p}\left(A_1^h, A_2^h, \ldots\ldots A_n^h\right)^{(0)} \,.\, \left|A_1^{a+h} \ldots\ldots A_{n-1}^{a+(n-1)h} A_n^{s+(n-p)h}\right),
\end{aligned}$$

a theorem which may be described as giving an expression for an alternant having two breaks in its series of indices in terms of alternants which have only one such break and that at the very last index. On account of the fact, however, that alternants of the latter kind are multiples of the alternant which has no break at all—that is to say, on account of the theorem

$$\left[A_1^h, A_2^h, \ldots, A_n^h\right]^{(p)} \,.\, \left|A_1^{a+h} A_2^{a+2h} \ldots\ldots A_n^{a+nh}\right) \\
= \left|A_1^{a+h} A_2^{a+2h} \ldots\ldots A_{n-1}^{a+(n-1)h} A_n^{a+(n+p)h}\right)$$

above given as an important special case of the general theorem of the third chapter—substitutions may be made which will result in the appearance of the last mentioned simple alternant in every term. Consequently, if we divide by this alternant and put $s = a + (n+m)h$ we have the theorem

$$\frac{\left| A_1^{a+h} A_2^{a+2h} \ldots\ldots A_{p-1}^{a+(p-1)h} A_p^{a+(p+1)h} \ldots\ldots A_{n-1}^{a+nh} A_n^{a+(n+m)h} \right)}{\left| A_1^{a+h} A_2^{a+2h} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots A_n^{a+nh} \right)}$$

$$\begin{aligned} = \ & \left(A_1^h, A_2^h, \ldots, A_n^h\right)^{(n-p)} . \left[A_1^h, A_2^h, \ldots, A_n^h\right]^{(m)} \\ - \ & \left(A_1^h, A_2^h, \ldots, A_n^h\right)^{(n-p-1)} . \left[A_1^h, A_2^h, \ldots, A_n^h\right]^{(m+1)} \\ + \ & \left(A_1^h, A_2^h, \ldots, A_n^h\right)^{(n-p-2)} . \left[A_1^h, A_2^h, \ldots, A_n^h\right]^{(m+2)} \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (-)^{n-p} & \left(A_1^h, A_2^h, \ldots, A_n^h\right)^{0} . \left[A_1^h, A_2^h, \ldots, A_n^h\right]^{(m+n-p)} . \end{aligned}$$

Again starting from the same initial identity we obtain the analogous series

$$\begin{aligned} M_{p,s} \quad &+ M_{p-1,s-h} = N_{n-p+1} . A_{s-h} \\ -M_{p-1,s-h} &- M_{p-2,s-2h} = -N_{n-p+2} . A_{s-2h} \\ +M_{p-2,s-2h} &+ M_{p-3,s-3h} = +N_{n-p+3} . A_{s-3h} \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (-)^{p-1}M_{1,s-(p-1)h} &+ \quad 0 \quad = (-)^{p-1}N_n . A_{s-ph} , \end{aligned}$$

and ∴ by addition have

$$M_{p,s} = N_{n-p+1} . A_{s-h} - N_{n-p+2} . A_{s-2h} + \ldots . (-)^{p-1}N_n . A_{s-ph}$$

or

$$\begin{aligned} & \left\| A_1^{a+h} A_2^{a+2h} \ldots\ldots A_{p-1}^{a+(p-1)h} A_p^{a+(p+1)h} \ldots\ldots A_{n-1}^{a+nh} A_n^{s} \right) \\ = \ & \left(A_1^h, A_2^h, \ldots, A_n^h\right)^{(n-p+1)} . \left\| A_1^{a+h} A_2^{a+2h} \ldots . A_{n-1}^{a+(n-1)h} A_n^{s-h} \right) \\ - & \left(A_1^h, A_2^h, \ldots, A_n^h\right)^{(n-p+2)} . \left| A_1^{a+h} A_2^{a+2h} \ldots . A_{n-1}^{a+(n-1)h} A_n^{s-2h} \right) \\ + & \left(A_1^h, A_2^h, \ldots, A_n^h\right)^{(n-p+3)} . \left| A_1^{a+h} A_2^{a+2h} \ldots . A_{n-1}^{a+(n-1)h} A_n^{s-3h} \right) \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (-)^{p-1} & \left(A_1^h, A_2^h, \ldots, A_n^h\right)^{(n)} . \left| A_1^{a+h} A_2^{a+2h} \ldots . A_{n-1}^{a+(n-1)h} A_n^{s-ph} \right) ; \end{aligned}$$

so that by substituting as above for each of the alternants on the right and dividing both sides by $\left| A_1^{a+h} A_2^{a+2h} \ldots A_n^{a+nh} \right)$ there results the alternative theorem

$$\frac{\left\| A_1^{a+h} A_2^{a+2h} \cdots\cdots A_{p-1}^{a+(p-1)h} A_p^{a+(p+1)h} \cdots\cdots A_{n-1}^{a+nh} A_n^{a+(n+m)h} \right)}{\left\| A_1^{a+h} A_2^{a+2h} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots A_n^{a+nh} \right)}$$

$$\begin{aligned} = {} & \left(A_1^h, \ldots, A_n^h\right)^{(n-p+1)} \cdot \left[A_1^h, \ldots, A_n^h\right]^{(m-1)} \\ & - \left(A_1^h, \ldots, A_n^h\right)^{(n-p+2)} \cdot \left[A_1^h, \ldots, A_n^h\right]^{(m-2)} \\ & + \left(A_1^h, \ldots, A_n^h\right)^{(n-p+3)} \cdot \left[A_1^h, \ldots, A_n^h\right]^{(m-3)} \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & (-)^{p-1}\left(A_1^h, \ldots, A_n^h\right)^{(n)} \cdot \left[A_1^h, \ldots, A_n^h\right]^{(m-p)}. \end{aligned}$$

Lastly, attention is drawn to the case where $a=0$, $h=1$, $s=1$, and to a case where the order of the alternants is infinite, viz., to the fraction

$$\frac{\left| A_1^b, A_2^a A_3^{a+h} A_4^{a+2h} \cdots\cdots A_n^{a+(n-2)h} A_{n+1}^{a+nh} \cdots\cdots A_\infty^\infty \right)}{\left| A_1^a, A_2^{a+h} A_3^{a+2h} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots A_\infty^\infty \right)}$$

The fifth and last chapter (pp. 395–398) concerns the simplest form of alternant above met with, viz., that in which the indices proceed throughout by a common difference, the main proposition being regarding the resolvability of the alternant into binomial factors. The property with which Cauchy and almost all later writers start is thus that with which Schweins ends. The mode of proof is interesting from its farfetchedness and ingenuity, but need not be given in full generality or in the original notation: the case of $|a^0b^1c^2d^3|$ will suffice.

The first step, then, is to select a row, say the last, and express the alternant in terms of the elements of this row and their complementary minors. In this way we obtain

$$|a^0b^1c^2d^3| = d^3|a^0b^1c^2| - d^2|a^0b^1c^3| + d|a^0b^2c^3| - |a^1b^2c^3|.$$

Now each of the alternants on the right is expressible as a multiple of $|a^0b^1c^2|$ by means of the theorem above given regarding alternants with one break in the continuity of the equidifferent progression of their indices. Using this we obtain

$$\begin{aligned} |a^0b^1c^2d^3| &= \left\{d^3 - d^2(a,b,c)^1 + d(a,b,c)^2 - (a,b,c)^3\right\} \cdot |a^0b^1c^2|, \\ &= \left\{d^3 - d^2(a+b+c) + d(ab+ac+bc) - abc\right\} \cdot |a^0b^1c^2|, \\ &= (d-a)(d-b)(d-c) \cdot |a^0b^1c^2|, \end{aligned}$$

when there only remains to continue the selfsame process upon the alternant of lower order now reached.

It may be remarked in passing that the identity

$$|a^0b^1c^2d^3| = d^3|a^0b^1c^2| - d^2|a^0b^1c^3| + d|a^0b^2c^3| - |a^1b^2c^3| ,$$

which expresses the alternant in descending powers of d, when taken along with the identity known to Cauchy

$$|a^0b^1c^2d^3| = (d-c)(d-b)(d-a)(c-b)(c-a)(b-a)$$

the right side of which may likewise be arranged in descending powers of d, viz.,

$$\{d^3 - d^2(a+b+c) + d(ab+ac+bc) - abc\}(c-b)(c-a)(b-a) ,$$

may have been the means of suggesting to Schweins his theorem regarding alternants like $|a^0b^2c^3|$, $|a^0b^1c^3|$ which have one break in their series of indices. In other words, the order in which he gives his theorems was very probably not the order of discovery.

The remaining portion of the chapter is an investigation of the quotient of two alternants of infinite order, viz.,

$$\frac{\| \mathrm{B}^a \mathrm{A}_1^{a+h} \mathrm{A}_2^{a+2h} \ \ldots\ldots\ \mathrm{A}_{n-1}^{a+(n-1)h} \mathrm{A}_{n+1}^{a+nh} \ \ldots\ldots\ \mathrm{A}_\infty^\infty)}{\| \mathrm{A}_1^a \mathrm{A}_2^{a+h} \mathrm{A}_3^{a+2h} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \mathrm{A}_\infty^\infty)} .$$

SYLVESTER (1839).

[On derivation of coexistence: Part 1, Being the theory of simultaneous simple homogeneous equations. *Philos. Mag.*, xvi. pp. 37–43.]

As has been already shown, Sylvester's first approach to the subject of determinants was similar to Cauchy's, the bases of both being the outward resemblance of the two expressions

$$bc^2 + a^2c + ab^2 - a^2b - ac^2 - b^2c ,$$
$$b_1c_2 + a_2c_1 + a_1b_2 - a_2b_1 - a_1c_2 - b_2c_1 .$$

As the former is equal to

$$(c-b)(c-a)(b-a) \quad \text{or} \quad \mathrm{PD}(abc),$$

i.e., product of the differences of a, b, c, Sylvester denoted the other, viz., the determinant

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix},$$

by $\zeta\mathrm{PD}(abc)$, ζ being his sign for multiplication according to the law $a_r \cdot a_s = a_{r+s}$. Using this notation he rediscovered, as has also already been seen, Schweins' theorem regarding the multiplication of the alternant

$$|a^1b^2c^3d^4 \ldots|$$

by such symmetric functions as

$$(a+b+c+\ldots), \ (ab+ac+\ldots+bc+\ldots), \ \ldots\ldots,$$

his form of statement being

$$\zeta(\mathrm{S}_r(abc\ldots l) \cdot \zeta\mathrm{PD}(0abc\ldots l) = \zeta_{-r}\mathrm{PD}(0abc\ldots l),$$

where ζ_{-r} implies that after 'zeta-ic' multiplication the subscripts are all to be diminished by r.

His attempted generalisation of this theorem has likewise been spoken of, its validity, however, being left undecided upon. Instead of the multiplier $\mathrm{S}_r(abc\ldots l)$ he proposed to take *any symmetric function whatever* of $a, b, c, \ldots, l$,—or, rather, *any function whatever followed by any symmetric function.* This would have been a most noteworthy extension which Schweins had not foreseen, but unfortunately there are grave doubts as to the truth of it,—indeed, one may go so far as to say that there would be no doubt whatever about the author's inaccuracy, were it not that there are doubts also as to his meaning. By way of test let us take the case where the multiplier of $|a^1b^2c^3d^4|$ is the symmetric function Σa^2bc. From later work* it is known that

$$|a^1b^2c^3d^4| \cdot \Sigma a^2b^1c^1d^0 = |a^1b^3c^4d^6| - 3|a^2b^3c^4d^5|,$$

whereas, according to Sylvester, there ought to be on the left only one alternant. Now although we know that Sylvester was in the habit of making guesses, and that these guesses though often brilliant were not always so,† it would be next to impossible to

* See Muir, "Theory of Determinants," p. 176 (1882).

† See *Crelle's Journal*, lxxxix. pp. 82–85.

find a generalisation of his which had no individual instances in support of it. There thus remains the curious and interesting question as to what amount of truth there is in the theorem as enunciated, and whether an amendment of the enunciation would not give something not merely unexceptionable but of important value.

In trying to pass from symmetric functions like Σa, Σab, Σabc, . . . which are linear in regard to each of the variables, and to extend the theorem to *any* symmetric function, Sylvester probably thought—at least it would be quite natural for him to do so—of expressing the latter in terms of the former and then applying the theorem already obtained. It is desirable, therefore, to see what such a process may lead to. Taking the case of the multiplier Σa^2bc we have

$$\begin{aligned} |a^1b^2c^3d^4|\,.\,\Sigma a^2bc &= |a^1b^2c^3d^4|\,.\,\{\Sigma a\,.\,\Sigma abc - 4\Sigma abcd\}\,, \\ &= \{|a^1b^2c^3d^4|\,.\,\Sigma a\}.\Sigma abc \;-\; |a^1b^2c^3d^4|.4\Sigma abcd\,, \\ &= |a^1b^2c^3d^5|\,.\,\Sigma abc \;-\; 4|a^2b^3c^4d^5|\,. \end{aligned}$$

At this point we encounter a difficulty, for the previous theorem, although it teaches us to multiply $|a^1b^2c^3d^4|$ by Σab, does not help us in the case where the multiplicand is $|a^1b^2c^3d^5|$. Proceeding, however, with other assistance we find the product

$$\begin{aligned} &= |a^2b^3c^4d^5| + |a^1b^3c^4d^6| - 4|a^2b^3c^4d^5|\,, \\ &= |a^1b^3c^4d^6| - 3|a^2b^3c^4d^5|\,, \end{aligned}$$

agreeing of course with what has already been found. Now the difficulty referred to would present itself to Sylvester also, but in a slightly different form by reason of the periodicity which he assumes in the elements. Thus, instead of writing

$$\begin{aligned} \{|a^1b^2c^3d^4|.\Sigma a\}\Sigma abc &= |a^1b^2c^3d^5\,|.\,\Sigma abc\,, \\ &= |a^2b^3c^4d^5| + |a^1b^3c^4d^6|\,, \end{aligned}$$

he would write

$$\zeta\{\zeta\mathrm{PD}(0abcd).\mathrm{S}_1(abcd)\}.\mathrm{S}_3(abcd) = \zeta\{\zeta_{-1}\mathrm{PD}(0abcd).\mathrm{S}_3(abcd)\}$$

and there pause for a little, not having specifically provided for the 'zeta-ic' multiplication of such an expression as $\zeta_{-1}\mathrm{PD}(0abcd)$ by

$S_3(abcd)$. The result forced upon him, however, would be the single term

$$\zeta_{-4}\mathrm{PD}(0abcd),$$

which in modern notation is

$$|a^2b^3c^4d^5|.$$

In the course of the work, therefore, the term $|a^1b^3c^4d^6|$ would be dropped altogether out of sight. The cause of this is undoubtedly the imposition of the condition just mentioned;—indeed, if we take the result of the work as above performed in the modern notation, viz. :—

$$|a^1b^3c^4d^6| - 3|a^2b^3c^4d^5|,$$

and make the elements periodic, *i.e.*, make

$$a^6,b^6,c^6,d^6 = a^1,b^1,c^1,d^1,$$

the first alternant will vanish by reason of having two indices alike, and we shall be left with a result agreeing with Sylvester's.

The conclusion, therefore, which we are tempted to draw is that if Sylvester's general theorem be correct it is only when the elements are subjected to periodicity.

JACOBI (1841).

[De functionibus alternantibus earumque divisione per productum e differentiis elementorum conflatum. *Crelle's Journ.*, xxii. pp. 360–371.]

After having treated of determinants in general (pp. 285–318), and of the special form which afterwards came to bear his own name (pp. 319–359), Jacobi turned to another special form which he had learned about from his great predecessor Cauchy. As, however, he differed from Cauchy in his mode of defining a determinant, Cauchy's definition, which, it will be remembered,

made use of the difference-product, now appears as a theorem and with it Jacobi makes his start; that is to say, he proves that

If in the determinant

$$\Sigma \pm \mathrm{a}_0\mathrm{b}_1\mathrm{c}_2\mathrm{d}_3 \ldots \mathrm{l}_{\mathrm{n}-1}$$

the suffixes be changed into exponents of powers, the result obtained is equal to the product of the $\frac{1}{2}\mathrm{n}(\mathrm{n}-1)$ *differences of* a, b, c, . . ., l, *viz., the product*

$$\begin{array}{r} (\mathrm{b}-\mathrm{a})(\mathrm{c}-\mathrm{a})(\mathrm{d}-\mathrm{a}) \;.\;.\;.\;.\; (\mathrm{l}-\mathrm{a}) \\ (\mathrm{c}-\mathrm{b})(\mathrm{d}-\mathrm{b}) \;.\;.\;.\;.\; (\mathrm{l}-\mathrm{b}) \\ (\mathrm{d}-\mathrm{c}) \;.\;.\;.\;.\; (\mathrm{l}-\mathrm{c}) \\ .\;.\;.\;.\;.\;.\;.\;. \end{array}$$

With the help of Sylvester's notation, which symbolizes the opposite change, viz., from exponents of powers to suffixes, this may be expressed in the compact form

$$\zeta\mathrm{PD}(abc \ldots l) = \Sigma \pm a_0b_1c_2 \ldots l_{n-1}.$$

In proving it he takes for granted (1) that *the product in question merely changes sign on the interchange of any two of the elements,* and (2) that *in the development of any function of this character there can be no term in which two or more exponents are equal,* for the reason that, if there were one such, there must be another exactly like it but of the opposite sign. Combining with this latter—which includes of course the case where the index 0 is repeated—the fact that, for the particular function under consideration, the indices must all be + and the sum of them equal to $\frac{1}{2}n(n-1)$, he concludes that no term can have any other indices than

$$0, 1, 2, \ldots, n-1.$$

Next, as there is only one way of getting an element, k say, in the $(n-1)^{\text{th}}$ power, viz., by multiplying all the $n-1$ binomial factors $k-a$, $k-b$, . . . in which k occurs, and after that only one way of getting an element, h say, in the $(n-2)^{\text{th}}$ power, viz., by taking from out the remaining binomial factors all the $n-2$ factors in which h occurs, and so on, it is inferred that no term can have any other coefficient than $+1$ or -1. Summing up

rather hurriedly, he consequently finds that the development of the product may be got by permuting in every possible way the indices of the term

$$a^0 b^1 c^2 \ldots l^{n-1}$$

and determining the signs in accordance with the law that the interchange of any pair causes the aggregate of all the terms to pass into the opposite value. This being exactly the mode of formation of the determinant $\Sigma \pm a_0 b_1 c_2 \ldots l_{n-1}$ with the difference that suffixes take the place of exponents of powers, the theorem is held to be established (. . . . "signis insuper ea lege definitis ut binorum indicum commutatione Aggregatum omnium terminorum in valorem oppositum abeat. Quæ ipsa est Determinantis formatio, siquidem exponentes pro indicibus habentur").

In passing, he remarks on the large number of vanishing terms in the development of the product, viz., $2^{\frac{1}{2}n(n-1)} - n!$, and the consequent desirability of obtaining this development from that of the determinant and not *vice versa.*

The fundamental relation between the determinant $\Sigma \pm a_0 b_1 c_2 \ldots l_{n-1}$ and the product of the differences of $a, b, c, \ldots, l$ having been established, it is then sought to find properties of the latter from the known properties of the former. What properties of the determinant are used Jacobi does not mention, all that is given being a bare enunciation of the results. It may be as well, however, to point out at once that all of them flow from one general theorem, viz., that of Laplace regarding the expansion of a determinant in terms of products of its minors.

The first is indicated by using as examples the case of three elements, a_1, a_2, a_3, and the case of four elements, a_1, a_2, a_3, a_4, viz.,

$$\begin{aligned}(a_2 - a_1)(a_3 - a_1)(a_3 - a_2) = \; & a_2 a_3 (a_3 - a_2) \\ & + a_3 a_1 (a_1 - a_3) \\ & + a_1 a_2 (a_2 - a_1),\end{aligned}$$

$$\begin{aligned}(a_2 - a_1)(a_3 - a_1) \ldots \ldots (a_4 - a_3) = \; & a_2 a_3 a_4 (a_3 - a_2)(a_4 - a_2)(a_4 - a_3) \\ & - a_3 a_4 a_1 (a_4 - a_3)(a_1 - a_3)(a_1 - a_4) \\ & + a_4 a_1 a_2 (a_1 - a_4)(a_2 - a_4)(a_2 - a_1) \\ & - a_1 a_2 a_3 (a_2 - a_1)(a_3 - a_1)(a_3 - a_2),\end{aligned}$$

it being pointed out that any term of the expansion is got from the preceding by cyclical permutation of the suffixes, and that the signs are all + when the number of elements is odd, and alternately + and − when the number of elements is even. The case of Laplace's expansion-theorem, which is here used, is easily seen to be that where the orders of the minors are $n-1$ and 1. Thus using later notation, we have

$$\zeta^{\frac{1}{2}}(abcd) = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix},$$

$$= |b^1c^2d^3| - |a^1c^2d^3| + |a^1b^2d^3| - |a^1b^2c^3|,$$

$$= bcd|b^0c^1d^2| - acd|a^0c^1d^2| + abd|a^0b^1d^2| - abc|a^0b^1c^2|,$$

which is the desired result.

In connection with this, it is perhaps worth noting that the result being, by the same case of Laplace's theorem, also equal to

$$-\begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix},$$

we may view Jacobi's first theorem as being equivalent to one of later date, viz.—

$$\zeta^{\frac{1}{2}}(a_1a_2a_3\ldots a_n) = (-)^{n-1}\begin{vmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{n-2} & a_2a_3a_4\ldots a_n \\ 1 & a_2 & a_2^2 & \ldots & a_2^{n-2} & a_1a_3a_4\ldots a_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_n & a_n^2 & \ldots & a_n^{n-2} & a_1a_2a_3\ldots a_{n-1} \end{vmatrix}.$$

When the determinant is of even order, it is possible to use that case of Laplace's expansion-theorem in which all the minors are of the 2nd order. Thus

$$\zeta^{\frac{1}{2}}(abcd) = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix},$$

$$= \begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix} \cdot \begin{vmatrix} c^2 & c^3 \\ d^2 & d^3 \end{vmatrix} - \begin{vmatrix} 1 & a \\ 1 & c \end{vmatrix} \cdot \begin{vmatrix} b^2 & b^3 \\ d^2 & d^3 \end{vmatrix} + \begin{vmatrix} 1 & a \\ 1 & d \end{vmatrix} \cdot \begin{vmatrix} b^2 & b^3 \\ c^2 & c^3 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & b \\ 1 & c \end{vmatrix} \cdot \begin{vmatrix} a^2 & a^3 \\ d^2 & d^3 \end{vmatrix} - \begin{vmatrix} 1 & b \\ 1 & d \end{vmatrix} \cdot \begin{vmatrix} a^2 & a^3 \\ c^2 & c^3 \end{vmatrix} - \begin{vmatrix} 1 & c \\ 1 & d \end{vmatrix} \cdot \begin{vmatrix} a^2 & a^3 \\ b^2 & b^3 \end{vmatrix},$$

$$= (b-a)(d-c)c^2d^2 - (c-a)(d-b)b^2d^2 + (d-a)(c-b)b^2c^2$$
$$+ (c-b)(d-a)a^2d^2 - (d-b)(c-a)a^2c^2 + (d-c)(b-a)a^2b^2,$$

$$= (b-a)(d-c)\{a^2b^2 + c^2d^2\}$$
$$+ (c-a)(b-d)\{a^2c^2 + d^2b^2\}$$
$$+ (d-x)(c-b)\{a^2d^2 + b^2c^2\}.$$

By Jacobi, however, the result here established is given merely as an example of an improved general theorem, which is enunciated in the form of a 'rule,' as follows :—

"Fingatur expressio

$$(a_1 - a_0)(a_3 - a_2) \ldots (a_n - a_{n-1})\Sigma \overset{2}{a_2}\overset{2}{a_3}\overset{4}{a_4}\overset{4}{a_5} \ldots \overset{n-1}{a_{n-1}}\overset{n-1}{a_n}$$

"quam quo clarius lex appareat sic scribam

$$(a_1 - a_0)(a_3 - a_2)\ldots(a_n - a_{n-1})\Sigma(a_0a_1)^0(a_2a_3)^2(a_4a_5)^4\ldots(a_{n-1}a_n)^{n-1},$$

"sub signo Σ omnimodis permutatis exponentibus

$$0, 2, 4, \ldots, n-1.$$

"In expressione illa cyclum percurrant *primo* elementa tria

$$a_{n-2}, a_{n-1}, a_n,$$

"*secundo* elementa quinque

$$a_{n-4}, a_{n-3}, a_{n-2}, a_{n-1}, a_n,$$

"et sic deinceps ita ut *postremo* cyclum percurrant elementa

$$a_1, a_2, a_3, \ldots, a_n.$$

"Omnium expressionum provenientium aggregatum æqua- "bitur ipsi P."

The meaning will be made quite apparent by taking a case other than Jacobi's above referred to, say the case where there are *six* elements, $a_0, a_1, a_2, \ldots, a_5$. According to the rule, what we have got to do at the outset is to form the term

$$(a_1 - a_0)(a_3 - a_2)(a_5 - a_4)\Sigma(a_0a_1)^0(a_2a_3)^2(a_4a_5)^4 ;$$

then derive from it two others by the cyclical substitution

$$\begin{pmatrix} a_3 & a_4 & a_5 \\ a_4 & a_5 & a_1 \end{pmatrix};$$

and finally, from each of these three derive four others by the cyclical substitution

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 & a_5 & a_1 \end{pmatrix}.$$

This being done, the sum of the fifteen terms so obtained can be taken as an expansion of the difference-product of $a_0, a_1, a_2, \ldots, a_5$.

Although, as has been said, the theorem is given without proof, it has to be noted that Jacobi draws attention to the fact that the number of ultimate terms in the expansion of the compound term

$$(a_1 - a_0)(a_3 - a_2)\ldots(a_n - a_{n-1})\Sigma(a_0a_1)^0(a_2a_3)^2(a_4a_5)^4\ldots(a_{n-1}a_n)^{n-1}$$

is

$$2^{\frac{n+1}{2}}.\left(1.2.3 \ldots . \frac{n+1}{2}\right):$$

that the number of ultimate terms obtainable from all the compound terms of this form is

$$2^{\frac{n+1}{2}}.\left(1.2.3 \ldots . \frac{n+1}{2}\right).(3.5 \ldots n):$$

and finally that this is equal to

$$1.2.3 \ldots (n+1),$$

a result which agrees with what we know of the difference-product from its determinant form.

From this general theorem regarding the difference-product of an even number of elements, an advance is made to a theorem of still greater generality, the means employed in obtaining it being

in all probability the same as before, viz., Laplace's expansion-theorem. The most general form of the latter theorem, it will be remembered, gives an expansion in terms of products of more than two minors. Jacobi was familiar with this, for in his famous fundamental memoir regarding general determinants a whole page (pp. 298, 299) is devoted to an illustration of it. Now, if we take the case where the number of minors is three, and apply it to the determinant which is the equivalent of the difference-product, we obtain a result which is transformable without difficulty into

$$\Pi(a_0, a_1, \ldots, a_n)$$
$$= \sum \pm \left\{ \begin{array}{c} (a_{i+1}a_{i+2}\ldots a_k)^{i+1}(a_{k+1}a_{k+2}\ldots a_n)^{k+1} \\ \times\, \Pi(a_0, a_1, \ldots, a_i)\Pi(a_{i+1}a_{i+2}\ldots a_k)\Pi(a_{k+1}a_{k+2}\ldots a_n) \end{array} \right\};$$

and this is the theorem "of still greater generality" above referred to.

Jacobi then proceeds to the consideration of alternating functions in general.

The definition which he gives, and to which he attaches Cauchy's name, is somewhat different from Cauchy's, being to the effect that *an alternating function is one which, by permutation of its variables, is either not changed at all, or is changed only in sign.*

In the matter of notation he also introduces a variation, but this time with more success. It will be remembered that, when Cauchy denoted a determinant by prefixing S ± to the typical term, he was simply following his practice in regard to alternating functions in general, which he denoted by

$$S \pm \phi(a,b,c, \ldots, l),$$

the rule for determining the sign of any term of the aggregate being left unexpressed. Instead of this, Jacobi uses

$$P \sum \left(\frac{\phi(a,b,c, \ldots, l)}{P} \right),$$

where P stands for the product of the differences of $a, b, c, \ldots, l$; and as the P which is inside the brackets is subject to permutation of its variables, and therefore automatically, as it were, changes sign with every interchange of a pair of variables, while the P which is outside the brackets remains unaltered, it is clear that

the rule of signs is here fully expressed. Thus, if $\phi(a,b,c, \ldots, l)$ were ab^2c^4, we should have

$$\Sigma\left(\frac{a^1b^2c^4}{\mathrm{P}}\right) = \frac{a^1b^2c^4}{(b-a)(c-a)(c-b)} + \frac{a^1c^2b^4}{(c-a)(b-a)(b-c)}$$

$$+ \frac{b^1a^2c^4}{(a-b)(c-b)(c-a)} + \frac{b^1c^2a^4}{(c-b)(a-b)(a-c)}$$

$$+ \frac{c^1a^2b^4}{(a-c)(b-c)(b-a)} + \frac{c^1b^2a^4}{(b-c)(a-c)(a-b)},$$

$$= \frac{a^1b^2c^4 - a^1c^2b^4 - b^1a^2c^4 + b^1c^2a^4 + c^1a^2b^4 - c^1b^2a^4}{(b-a)(c-a)(c-b)},$$

$$\text{and} \therefore \mathrm{P}\Sigma\left(\frac{a^1b^2c^4}{\mathrm{P}}\right) = a^1b^2c^4 - a^1c^2b^4 - b^1a^2c^4 + b^1c^2a^4 + c^1a^2b^4 - c^1b^2a^4,$$

which is an alternating function written by Cauchy in the form $S(\pm a^1b^2c^4)$, and which, being a determinant, was written by Jacobi himself also in the form $\Sigma \pm a^1b^2c^4$.

It is pointed out that any term of ϕ which remains unchanged by the interchange of two of the variables may be left out of account; but the question raised by Cauchy regarding possible and impossible forms of ϕ is not touched upon. As a corollary, it is stated that if

$$\phi(a_0,a_1, \ldots, a_n) = a_0^{\alpha_0}a_1^{\alpha_1} \ldots a_n^{\alpha_n},$$

the indices $\alpha_0, \alpha_1, \ldots, \alpha_n$ must be all different if the alternating function is not to vanish.

He then recalls the known fact that, when the indices $\alpha_0, \alpha_1, \ldots, \alpha_n$ are integral, the alternating function

$$\Sigma \pm a_0^{\alpha_0}a_1^{\alpha_1} \ldots a_n^{\alpha_n} \quad \text{or} \quad \mathrm{P}\Sigma\frac{a_0^{\alpha_0}a_1^{\alpha_1} \ldots a_n^{\alpha_n}}{\mathrm{P}}$$

is divisible by P, the difference-product of $a_0, a_1, \ldots, a_n$, and puts to himself the problem of finding the generating function of the quotient

$$\Sigma\frac{a_0^{\alpha_0}a_1^{\alpha_1} \ldots a_n^{\alpha_n}}{\mathrm{P}}.$$

In the course of this quest his first proposition is—

If ϕ be any rational integral function of $\mathrm{m}+1$ *variables,* Π *their difference-product, and* f *be a function of the* $(\mathrm{n}+1)^{\mathrm{th}}$ *degree in one variable and be of the form* $(\mathrm{x}-\mathrm{a}_0)(\mathrm{x}-\mathrm{a}_1)\ldots\ldots(\mathrm{x}-\mathrm{a}_\mathrm{n})$, *then when* $\mathrm{m}>\mathrm{n}$ *no single term of the expansion of*

$$\frac{\Pi(\mathrm{t}_0,\mathrm{t}_1,\ldots,\mathrm{t}_\mathrm{m})\phi(\mathrm{t}_0,\mathrm{t}_1,\ldots,\mathrm{t}_\mathrm{m})}{\mathrm{f}(\mathrm{t}_0)\mathrm{f}(\mathrm{t}_1)\ldots\ldots\mathrm{f}(\mathrm{t}_\mathrm{m})},$$

according to descending powers of $\mathrm{t}_0, \mathrm{t}_1, \ldots, \mathrm{t}_\mathrm{m}$, *can contain negative powers of all these variables.*

To prove it, he of course uses the identity

$$\frac{1}{f(x)} \quad i.e., \quad \frac{1}{(x-a_0)(x-a_1)\ldots\ldots(x-a_m)}$$

$$=\frac{1}{f'(a_0).(x-a_0)}+\frac{1}{f'(a_1).(x-a_1)}+\ldots\ldots+\frac{1}{f'(a_m).(x-a_m)},$$

and thus changes the expression into the form

$$\Pi\phi\left\{\frac{1}{f'(a_0).(t_0-a_0)}+\frac{1}{f'(a_1).(t_0-a_1)}+\ldots\ldots+\frac{1}{f'(a_m).(t_0-a_m)}\right\}$$

$$\times\left\{\frac{1}{f'(a_0).(t_1-a_0)}+\frac{1}{f'(a_1).(t_1-a_1)}+\ldots\ldots+\frac{1}{f'(a_n).(t_1-a_n)}\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\times\left\{\frac{1}{f'(a_0).(t_m-a_0)}+\frac{1}{f'(a_1).(t_m-a_1)}+\ldots\ldots+\frac{1}{f'(a_n).(t_m-a_n)}\right\}.$$

He then says that the result of performing the multiplication of these bracketed factors is to produce terms of the form

$$\frac{\Pi\phi}{f'(a)f'(b)\ldots f'(p).(t_0-a)(t_1-b)\ldots(t_m-p)},$$

where each of the $m+1$ quantities $a, b, \ldots, p$ is necessarily one of the $n+1$ quantities $a_0, a_1, \ldots, a_n$, and where, therefore, on account of m being greater than n, the quantities $a, b, \ldots, p$ cannot be all different. But terms of this form can be changed into

$$\frac{\phi}{f'(a)f'(b)\ldots f'(p)}\cdot\frac{\Pi}{t_1-b-t_0+a}\left\{\frac{1}{t_0-a}-\frac{1}{t_1-b}\right\}\cdot\frac{1}{(t_2-c)(t_3-d)\ldots(t_m-p)},$$

which shows that in the case of two of the quantities $a, b, \ldots, p$ being alike, say a and b, the second factor would become

$$\frac{\Pi}{t_1-t_0},$$

and therefore could be simplified by having $t_1 - t_0$ struck out of both numerator and denominator. This means that when $m > n$ the second factor, like the first, can have only positive integral powers of the variables. As for the third and fourth factors, their product is the difference of the two fractions

$$\frac{1}{(t_0-a)(t_2-c)(t_3-d)\ldots(t_m-p)} \text{ and } \frac{1}{(t_1-a)(t_2-c)(t_3-d)\ldots(t_m-p)},$$

the former of which yields no negative powers of t_1, and the latter no negative powers of t_0. The proposition is thus established.

To prove the next proposition he utilizes the theorem that

If F *be any rational integral function of a number of variables, the coefficient of* $\mathrm{x}^{-1}\mathrm{y}^{-1}\mathrm{z}^{-1}$ . . . . *in the expansion of*

$$\frac{\mathrm{F}(\mathrm{x},\mathrm{y},\mathrm{z},\ \ldots)}{(\mathrm{x}-\mathrm{a})(\mathrm{y}-\mathrm{b})(\mathrm{z}-\mathrm{c})\ \ldots}$$

according to descending powers of x, y, z, ... *is*

$$\mathrm{F}(\mathrm{a},\mathrm{b},\mathrm{c},\ \ldots).$$

This is spoken of as being well-known, and no proof of it is given. It is readily seen, however, that as the expansion referred to is got by performing the multiplications indicated in

$$\begin{aligned} \mathrm{F}(x,y,z,\ \ldots)\ .\ &\{x^{-1}+ax^{-2}+a^2x^{-3}+\ \ldots\} \\ &\{y^{-1}+by^{-2}+b^2y^{-3}+\ \ldots\} \\ &\{z^{-1}+cz^{-2}+c^2z^{-3}+\ \ldots\} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

any term in F, say the term $\mathrm{A}x^\alpha y^\beta z^\gamma$ . . . . , would require to be multiplied by $x^{-\alpha-1}$, $y^{-\beta-1}$, $z^{-\gamma-1}$, . . . in order to produce a term in $x^{-1}y^{-1}z^{-1}$ . . . ., and that these multipliers being only found associated with the coefficients a^α, b^β, c^γ, . . . the term so produced would have for its coefficient $\mathrm{A}a^\alpha b^\beta c^\gamma$ . . . . The full coefficient of $x^{-1}y^{-1}z^{-1}$ . . . . would thus be $\mathrm{F}(a,b,c,\ \ldots)$.

He also uses an identity regarding difference-products which it may be as well to state separately, viz., that

$$\begin{aligned} &\Pi(a_0,a_1,\ldots,a_n)\ .\ \Pi(a_{n-m},a_{n-m+1},\ldots,a_n) \\ &\quad = (-1)^{\frac{1}{2}m(m+1)}\Pi(a_0,a_1,\ldots,a_{n-m-1})\ .\ f'(a_{n-m})f'(a_{n-m+1})\ldots f'(a_n), \end{aligned}$$

where $f'(a_r)$ stands for the product of the n factors got by subtracting from a_r each of the quantities $a_0, a_1, \ldots, a_n$ except a_r.

This he holds to be true,* because the product

$$f'(a_{n-m})f'(a_{n-m+1}) \;.\;.\;.\;.\; f'(a_n)$$

contains as factors the differences of all the elements $a_0, a_1, \ldots, a_n$ except those which go to make $\Pi(a_0, a_1, \;.\;.\;.\;.\; a_{n-m-1})$ and contains a second time but with opposite signs the $\frac{1}{2}m(m+1)$ factors which go to make $\Pi(a_{n-m}, a_{n-m+1}, \ldots, a_n)$.

* The factors of a difference-product may always be, and usually are, arranged in the form of a right-angled isosceles triangle: for example,

$$\begin{array}{rr} \zeta^{\frac{1}{2}}(abcdefg) = & (b-a)(c-a)(d-a)(e-a)(f-a)(g-a) \\ & (c-b)(d-b)(e-b)(f-b)(g-b) \\ & (d-c)(e-c)(f-c)(g-c) \\ & (e-d)(f-d)(g-d) \\ & (f-e)(g-e) \\ & (g-f)\,. \end{array}$$

Consequently there must be an algebraic identity corresponding to the geometrical proposition—*If from a point in the hypotenuse of an isosceles right-angled triangle straight lines be drawn parallel to the other sides, the triangle is thereby divided into two triangles of the same kind and a rectangle.* This identity it is which is at the basis of Jacobi's, for drawing the lines thus—

$$\begin{array}{r:r} (b-a)(c-a)(d-a) & (e-a)(f-a)(g-a) \\ (c-b)(d-b) & (e-b)(f-b)(g-b) \\ (d-c) & (e-c)(f-c)(g-c) \\ & (e-d)(f-d)(g-d) \\ \hdashline & (f-e)(g-e) \\ & (g-f)\,, \end{array}$$

we obtain

$$\begin{array}{rr} \zeta^{\frac{1}{2}}(abcdefg) = \zeta^{\frac{1}{2}}(abcd)\,.\,\zeta^{\frac{1}{2}}(efg)\,. & (e-a)(f-a)(g-a) \\ & (e-b)(f-b)(g-b) \\ & (e-c)(f-c)(g-c) \\ & (e-d)(f-d)(g-d)\,. \end{array}$$

But the expression here which corresponds to the rectangle in the geometrical proposition

$$\begin{aligned} &= \left.\begin{array}{lll} (e-a) & (f-a) & (g-a) \\ (e-b) & (f-b) & (g-b) \\ (e-c) & (f-c) & (g-c) \\ (e-d) & (f-d) & (g-d) \\ \;\;. & (f-e) & (g-e) \\ (e-f) & \;\;. & (g-f) \\ (e-g) & (f-g) & \;\;. \end{array}\right\} \div \zeta^{\frac{1}{2}}(efg)\,.\,\zeta^{\frac{1}{2}}(gfe) \\ &= f'(e)f'(f)f'(g) \div (-)^3\zeta^{\frac{1}{2}}(efg)\,.\,\zeta^{\frac{1}{2}}(efg)\,. \end{aligned}$$

Consequently

$$\frac{\zeta^{\frac{1}{2}}(abcdefg)\,.\,\zeta^{\frac{1}{2}}(efg)}{\zeta^{\frac{1}{2}}(abcd)} = (-)^3 f'(e)f'(f)f'(g),$$

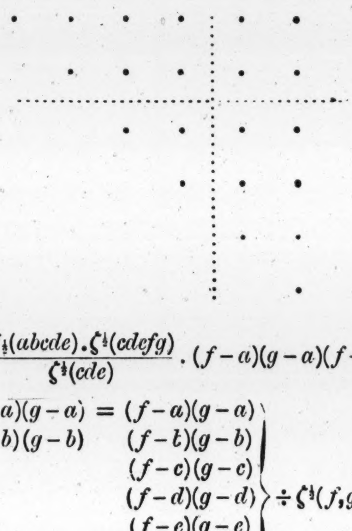
which is Jacobi's identity.

These preliminaries having been given, the second proposition may now be proceeded with. It is—

If ϕ *be any rational integral function of* $\mathrm{m}+1$ *variables,* Π *their difference-product, and* f *be a function of the* $(\mathrm{n}+1)^{\mathrm{th}}$ *degree in one variable and be of the form* $(\mathrm{x}-\mathrm{a}_0)(\mathrm{x}-\mathrm{a}_1)\ \ldots\ (\mathrm{x}-\mathrm{a}_\mathrm{n})$, *then when* $\mathrm{m} \not> \mathrm{n}$ *the coefficient of* $\mathrm{t}_0^{-1}\mathrm{t}_1^{-1}\ \ldots\ \mathrm{t}_\mathrm{m}^{-1}$ *in the expansion of*

$$\frac{\Pi(\mathrm{t}_0,\mathrm{t}_1,\ \ldots,\ \mathrm{t}_\mathrm{m})\phi(\mathrm{t}_0,\mathrm{t}_1,\ \ldots,\ \mathrm{t}_\mathrm{m})}{\mathrm{f}(\mathrm{t}_0)\mathrm{f}(\mathrm{t}_1)\ \ldots\ \mathrm{f}(\mathrm{t}_\mathrm{m})}$$

It is easily seen that there is still an analogue when the point through which the parallels are drawn is inside the triangle: thus, corresponding to the diagram



we have the identity

$$\zeta^{\frac12}(abcdefg) = \frac{\zeta_{\frac12}(abcde)\,.\,\zeta^{\frac12}(cdefg)}{\zeta^{\frac12}(cde)}\,.\,(f-a)(g-a)(f-b)(g-b),$$

and as

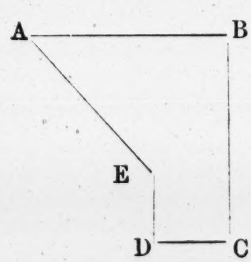
$$\begin{array}{l} (f-a)(g-a) \\ .\,(f-b)(g-b) \end{array} = \left.\begin{array}{c} (f-a)(g-a) \\ (f-b)(g-b) \\ (f-c)(g-c) \\ (f-d)(g-d) \\ (f-e)(g-e) \\ .\quad (g-f) \\ (f-g)\quad . \end{array}\right\} \div \zeta^{\frac12}(f,g)\zeta^{\frac12}(g,f)$$

$$= f'(f)\,f'(g) \div (-)^1\zeta^{\frac12}(f,g)\zeta^{\frac12}(f,g),$$

it follows that

$$\frac{\zeta^{\frac12}(abcdefg)\,.\,\zeta_{\frac12}(cde)}{\zeta^{\frac12}(abcde)\,.\,\zeta^{\frac12}(cdefg)} = (-)^1\frac{f'(f)\,f'(g)}{\zeta^{\frac12}(fg)\zeta^{\frac12}(fg)}\,.$$

It should be noticed, however, that the absolutely perfect geometrical analogue to Jacobi's identity is got by taking a rectilineal figure of the form ABCDE, where AB = BC, CD = DE, B = C = D = 90°, and then equating the sum of the two parts got by joining CE to the sum of the two parts got by producing DE to meet AB in F. Further, the exact analogue to his proof would be to say that the rectangle BCDF contains all of the triangle ABC except the triangle AEF, and contains the triangle CDE in addition.



according to descending powers of $t_0, t_1, \ldots, t_m$ *is*

$$(-)^{\frac{1}{2}m(m+1)}\sum\frac{a_0^0 a_1^1 a_2^2 \ldots a_{n-m-1}^{n-m-1}\phi(a_{n-m}, a_{n-m+1}, \ldots, a_n)}{\Pi(a_0, a_1, \ldots, a_n)}$$

effect being given to the sign of summation by permuting in every possible way the quantities $a_0, a_1, \ldots, a_n$.

As has already been seen the expression to be expanded is equal to an aggregate of terms of the form

$$\frac{\phi(t_0, t_1, \ldots, t_m) \cdot \Pi(t_0, t_1, \ldots, t_m)}{f'(a)f'(b) \ldots f'(p) \cdot (t_0 - a)(t_1 - b) \ldots (t_m - p)},$$

where each of the $m+1$ quantities $a, b, \ldots, p$ is one of the $n+1$ quantities $a_0, a_1, \ldots a_n$. Since, however, we are now in search of the coefficient of $t_0^{-1}t_1^{-1} \ldots t_m^{-1}$ we may leave out of account all terms of this aggregate which have two or more of the $m+1$ quantities $a, b, \ldots, p$ alike, for it has been shown that the expansion of such a term cannot contain $t_0^{-1}t_1^{-1} \ldots t_m^{-1}$. We are thus left with an aggregate which may be represented by

$$S\frac{\phi(t_0, t_1, \ldots, t_m) \cdot \Pi(t_0, t_1, \ldots, t_m)}{f'(a_{n-m})f'(a_{n-m+1}) \ldots f'(a^n) \cdot (t_0 - a_{n-m})(t_1 - a_{n-m+1}) \ldots (t_m - a_n)},$$

it being understood that for $a_{n-m}, a_{n-m+1}, \ldots, a_n$ is to be taken any permutation of $m+1$ quantities of the group $a_0, a_1, \ldots, a_n$. But, if the coefficient of $t_0^{-1}t_1^{-1} \ldots t_m^{-1}$ in this be denoted by H, we have by the first of our auxiliary theorems

$$H = S\frac{\phi(a_{n-m}, a_{n-m+1}, \ldots, a_n) \cdot \Pi(a_{n-m}, a_{n-m+1}, \ldots, a_n)}{f'(a_{n-m})f'(a_{n-m+1}) \ldots f'(a^n)},$$

and using the second to substitute

$$(-1)^{\frac{1}{2}m(m+1)}\Pi(a_0, a_1, \ldots, a_{n-m-1}) / \Pi(a_0, a_1, \ldots, a_n)$$

for $\Pi(a_{n-m}, a_{n-m+1}, \ldots, a_n) / f'(a_{n-m})f'(a_{n-m+1}) \ldots f'(a_n)$

we have

$$H = (-1)^{\frac{1}{2}m(m+1)}S\frac{\Pi(a_0, a_1, \ldots, a_{n-m-1}) \cdot \phi(a_{n-m}, a_{n-m+1}, \ldots, a_n)}{P},$$

where, be it remembered, the $n+1$ elements $a_0, a_1, \ldots, a_n$ are to be separated in every possible way into two classes containing

$n-m$ and $m+1$ elements respectively, and all permutations of the elements of the second class are to be taken. In this expression, however, another substitution can be made by reason of the identity

$$\frac{\Pi(a_0,a_1,\ldots,a_{n-m-1})}{\mathrm{P}} = \sum \frac{a_0^0 a_1^1 \ldots\ldots a_{n-m-1}^{n-m-1}}{\mathrm{P}},$$

where under the sign Σ all possible permutations of the indices $0, 1, \ldots, n-m-1$ are to be taken. When this substitution has been made, we shall consequently have to take every possible permutation of *both* classes of elements. But to take every possible separation into two classes and permute the elements of each of the classes in every possible way is the same as to take every possible permutation of *all* the elements. Our result will therefore be

$$\mathrm{H} = (-1)^{\frac{1}{2}m(m+1)}\sum \frac{a_0^0 a_1^1 \ldots a_{n-m-1}^{n-m-1} \cdot \phi(a_{n-m},a_{n-m+1},\ldots,a_n)}{\mathrm{P}},$$

if it be understood that under the sign of summation all possible permutations of $a_0, a_1, \ldots a_n$ are to be taken: and this is what we set out to prove.

The case where $m=n$ is then considered, because of its special interest. The first expression obtained above for H becomes in this case

$$\sum \frac{\mathrm{P}\,.\,\phi(a_0,a_1,\ldots,a_n)}{f'(a_0)f'(a_1)\ldots f'(a_n)},$$

where under Σ all permutations of $a_0, a_1, \ldots, a_n$ are to be taken. Making in this the substitution which is possible by reason of the identity

$$f'(a_0)f'(a_1)\ldots f'(a_n) = (-1)^{\frac{1}{2}n(n+1)}\mathrm{P}^2,$$

we have

$$\mathrm{H} = (-1)^{\frac{1}{2}n(n+1)}\mathrm{S}\frac{\phi(a_0,a_1,\ldots,a_n)}{\mathrm{P}}.$$

The formal enunciation of the result thus obtained is:—

If ϕ *be any rational integral function of* $n+1$ *variables,* Π *their difference-product, and* f *be a function of the* $(n+1)^{\text{th}}$ *degree in one*

variable and be of the form $(x-a_0)(x-a_1)\ .\ .\ .\ (x-a_n)$; *then the coefficient of* $t_0^{-1}t_1^{-1}\ .\ .\ .\ t_n^{-1}$ *in the expansion of*

$$(-1)^{\frac{1}{2}n(n+1)}\frac{\Pi(t_0,t_1,\ .\ .\ .,\ t_n)\cdot\phi(t_0,t_1,\ .\ .\ .,\ t_n)}{f(t_0)f(t_1)\ .\ .\ .\ f(t_n)}$$

is

$$\sum\frac{\phi(a_0,a_1,\ .\ .\ .,\ a_n)}{\Pi(a_0,a_1,\ .\ .\ .,\ a_n)},$$

effect being given to the sign of summation by permuting in every possible way the elements $a_0,\ a_1,\ .\ .\ .,\ a_n$.

As we have seen above that

$$\sum\frac{\phi(a_0,a_1,\ .\ .\ .,\ a_n)}{\Pi(a_0,a_1,\ .\ .\ .,\ a_n)}$$

is the quotient of any rational integral alternating function by the difference-product of its elements, and that this quotient is often in request, it is important for practical purposes to note that what this last theorem of Jacobi's gives is the generating function of the said quotient.

After giving a line or two to the case where $m=n-1$, Jacobi returns to the general theorem and specializes in another direction, viz., by putting

$$\phi(t_0,t_1,\ldots,t_m)=t_0^{\gamma}t_1^{\gamma_1}\ldots t_m^{\gamma_m}.$$

Division of both sides by ϕ is in this case possible, and the resulting theorem is one of considerable importance :—

The expression

$$\sum\frac{a_0^0a_1^1\ .\ .\ .\ a_{n-m-1}^{n-m-1}a_{n-m}^{\gamma}a_{n-m+1}^{\gamma_1}\ .\ .\ .\ .\ a_n^{\gamma_m}}{(a_1-a_0)(a_2-a_0)\ .\ .\ .\ (a_n-a_{n-1})}$$

which is the quotient of an alternating function by the difference-product of its elements is equal to the coefficient of

$$t_0^{-(\gamma+1)}t_1^{-(\gamma_1+1)}\ .\ .\ .\ t_m^{-(\gamma_m+1)}$$

in the expansion of

$$\frac{(t_0-t_1)(t_0-t_2)\ .\ .\ .\ (t_{n-1}-t_n)}{f(t_0)f(t_1)\ .\ .\ .\ f(t_m)}$$

according to descending powers of $t_0, t_1, \ldots, t_m$, *where*

$$f(x) = (x - a_0)(x - a_1) \ldots (x - a_n).$$

This is followed up by actually working out the expansion in question, the numerator being of course changed into

$$\Sigma \pm t_0^m t_1^{m-1} \ldots t_{m-1},$$

and its cofactor

$$\frac{1}{f(t_0)} \cdot \frac{1}{f(t_1)} \cdot \ldots \cdot \frac{1}{f(t_m)}$$

into

$$\left(\frac{1}{t_0^{n+1}} + \frac{C_1}{t_0^{n+2}} + \frac{C_2}{t_0^{n+3}} + \ldots + \frac{C_s}{t_0^{n+1+s}} + \ldots\right)$$

$$\times\left(\frac{1}{t_1^{n+1}} + \frac{C_1}{t_1^{n+2}} + \frac{C_2}{t_1^{n+3}} + \ldots + \frac{C_s}{t_0^{n+1+s}} + \ldots\right)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\times\left(\frac{1}{t_m^{n+1}} + \frac{C_1}{t_m^{n+2}} + \frac{C_2}{t_m^{n+3}} + \ldots + \frac{C_s}{t_m^{n+1+s}} + \ldots\right),$$

where C_s is the sum of all the products of s elements, different or equal, taken from $a_0, a_1, \ldots, a_n$. Multiplication of these $m+1$ partial factors has next to be performed, the general term of the result being seen to be

$$\frac{C_{s_0} C_{s_1} \ldots C_{s_m}}{t_0^{n+1+s_0} t_1^{n+1+s_1} \ldots t_m^{n+1+s_m}}.$$

All that remains, then, is the multiplication of this result by the corresponding expression for the original numerator, *i.e.*, by $\Sigma \pm t_0^m t_1^{m-1} \ldots t_{m-1}$, which, be it noted, consists of $(m+1)^2$ terms, the Σ referring to permutation of the indices $m, m-1, \ldots, 1, 0$. Without further delay, Jacobi merely adds that the general term will therefore become

$$\Sigma \pm \frac{C_{s_0} C_{s_1} \ldots C_{s_m}}{t_0^{n-m+1+s_0} t_1^{n-m+1+s_1} \ldots t_m^{n+1+s_m}},$$

and that consequently the proposition last formulated will 'suggest' the identity

$$\sum \frac{a_1^1 a_2^2 \cdots a_{n-m-1}^{n-m-1} a_{n-m}^{\gamma} a_{n-m+1}^{\gamma_1} \cdots a_n^{\gamma_m}}{(a_1-a_0)(a_2-a_0)\cdots(a_n-a_{n-1})}$$

$$= \sum \pm C_{\gamma+m-n} C_{\gamma_1+m-n-1} \cdots C_{\gamma_m-n},$$

where the Σ in the first case refers to permutation of $a_0, a_1, \ldots, a_n$, and in the second case to permutation of $\gamma, \gamma_1, \ldots, \gamma_m$. In a couple of lines it is next pointed out that the putting of $m=0$, $m=1$, . . . in this suggested identity gives

$$\sum \frac{a_1^1 a_2^2 \cdots a_{n-1}^{n-1} a_n^{\gamma}}{P} = C_{\gamma-n},$$

$$\sum \frac{a_1^1 a_2^2 \cdots a_{n-2}^{n-2} a_{n-1}^{\gamma} a_n^{\gamma_1}}{P} = C_{\gamma+1-n} C_{\gamma_1-n} - C_{\gamma_1+1-n} C_{\gamma-n},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots ;$$

then, rather unexpectedly, there is given a mere restatement of the identity itself, viz. :—

"Generaliter *æquatur quotiens propositus*

$$\sum \frac{a_1 a_2^2 \cdots a_{n-m-1}^{n-m-1} a_{n-m}^{\gamma} a_{n-m+1}^{\gamma_1} \cdots a_n^{\gamma_m}}{P}$$

determinanti quod pertinet ad systema quantitatum

$$\begin{array}{llll} C_{\gamma+m-n} & C_{\gamma_1+m-n} & \cdots & C_{\gamma_m+m-n} \\ C_{\gamma+m-n-1} & C_{\gamma_1+m-n-1} & \cdots & C_{\gamma_m+m-n-1} \\ \cdots & \cdots & \cdots & \cdots \\ C_{\gamma-n} & C_{\gamma_1-n} & \cdots & C_{\gamma_m-n}." \end{array}$$

This is the last result of the memoir, the few additional lines used being merely for the purpose of showing how the determinant just mentioned may be simplified. The simplification consists in leaving out the element a_n in forming the C's of the second row from the end, the elements a_n, a_{n-1} in forming the C's of the third row from the end, and so on. The reason in the first case is that this will have the same effect as subtracting from each element of the row a_n times the corresponding element of the last row, and the reason in other cases is similar. If C′ be

used to stand for the same as C, but to concern one element less, viz., a_n, and C″ be used in similar manner, the identities at the bottom of the simplification are—

$$C_{s+1} - a_n C_s = C'_{s+1},$$

$$C_{s+2} - (a_n + a_{n-1})C_{s+1} + a_n a_{n-1} C_s = C''_{s+2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

the truth of which is apparent when we remember that C_1, C_2, . . . are practically defined by the equation

$$\frac{1}{(x-a_0)(x-a_1)\ \ldots\ (x-a_n)} = \frac{1}{x^{n+1}} + \frac{C_1}{x^{n+2}} + \frac{C_2}{x^{n+3}} + \ldots$$

It is noted also that in the determinant a C with the suffix 0 is to be taken as 1, and a C with a negative suffix as 0.

CAUCHY (1841).

[Mémoire sur les fonctions alternées et sur les sommes alternées. *Exercises d'Analyse*, ii, pp. 151–159.]

As has before been pointed out, the preceding paper of Jacobi's was the last of a triad which was followed up by a similar triad from the pen of Cauchy. Cauchy's first paper, which corresponds in subject to Jacobi's third, comes up therefore quite appropriately for discussion now.

What is really new in the first part of it concerns the finding of the symmetric function which is the quotient of an alternating function by the difference-product of the elements; that is to say, in Cauchy's notation, the finding of

$$\frac{S[\pm f(x,y,z,\ \ldots)]}{(x-y)(x-z)\ \ldots\ (y-z)\ \ldots},$$

or, in Jacobi's notation, the finding of

$$\sum \frac{f(x,y,z,\ \ldots)}{\Pi(x,y,z,\ \ldots)}.$$

It therefore opens with the reminder:—

"Une fraction rationelle qui a pour dénominateur une fonction symétrique et pour numérateur une fonction alternée

des variables $x, y, z, \ldots$ est évidemment elle-même une fonction alternée de ces variables. Réciproquement, si une fonction alternée de $x, y, z, \ldots$ se trouve représentée par une fraction rationnelle dont le dénominateur se réduise à une fonction symétrique, le numérateur de la même fraction rationnelle sera nécessairement une autre fonction alternée de $x, y, z, \ldots$."

This prepares us for the consideration of the alternating aggregate

$$S[\pm f(x,y,z, \ldots)]$$

where f is fractional and rational, and where, although Cauchy does not explicitly say so, the numerator and denominator are integral. In regard to this he asserts that the various fractions which compose the aggregate may be combined into one fraction U/V, where V is an integral symmetric function divisible by all the denominators, and where, therefore, U will necessarily be an integral alternating function and, as such, be divisible by the difference-product of its variables. We are thus led to the proposition that the given alternating function of $x, y, z, \ldots$ can be resolved into two factors, one of which is the difference-product (P) of $x, y, z, \ldots$, and the other of the form W/V, where W and V are integral symmetric functions of the same variables.

As an illustration of this, full consideration is given to the case where

$$f(x,y,z, \ldots) = \frac{1}{(x-a)(y-b)(z-c) \ldots .},$$

the number of variables being n. The appropriate symmetric function V, which is divisible by all the denominators of the aggregate $\Sigma[\pm f(x,y,z, \ldots)]$ is evidently in this case

$$(x-a)(x-b)(x-c) \ldots . (y-a)(y-b)(y-c) \ldots . (z-a)(z-b)(z-c) \ldots .$$

or, say,

$$F(x) . F(y) . F(z) \ldots . . ;$$

and the corresponding numerator U, always divisible by the difference-product of $x, y, z, \ldots$ is in this case, because of the peculiar form * of the denominator of the function f, also divisible by the

* The form is such that the result of any interchange among $x, y, z, \ldots$ is attainable by a corresponding interchange among $a, b, c, \ldots .$

difference-product of $a, b, c, \ldots$ It is thus seen that the given alternating aggregate

$$\sum\left[\pm\frac{1}{(x-a)(x-b)(x-c)\ldots}\right]=k.\frac{\mathrm{PP}'}{\mathrm{V}},$$

where P,P′,V are known, and k has still to be found. An easy step further is made by inquiring as to the *degree* of k, it being noted in this connection that the degree on the one side is $-n$, and that on the other side the degree of $\mathrm{P}=\frac{1}{2}n(n-1)$, the degree of P′ likewise $=\frac{1}{2}n(n-1)$, and the degree of $\mathrm{V}=n^2$. The resultant degree of PP′/V on the right is therefore inferred to be

$$\begin{aligned}&=\tfrac{1}{2}n(n-1)+\tfrac{1}{2}n(n-1)-n^2,\\&=-n;\end{aligned}$$

and as a consequence the degree of k must be zero. In other words, k must be constant in regard to $x, y, z, \ldots, a, b, c, \ldots$: so that for its full determination the best thing to do is to select as easy a special case as possible. Cauchy's choice falls on the case where $x=a$, $y=b$, $z=c$, $\ldots$; and preparatory for this substitution he transforms the above result,

$$\sum\left[\pm\frac{1}{(x-a)(y-b)(z-c)\ldots}\right]=k.\frac{\mathrm{PP}'}{\mathrm{V}},$$

into

$$\begin{aligned}k.\mathrm{PP}'&=\mathrm{V}.\sum\left[\pm\frac{1}{(x-a)(y-b)(z-c)\ldots}\right],\\&=\sum\left[\pm\frac{\mathrm{V}}{(x-a)(y-b)(z-c)\ldots}\right].\end{aligned}$$

As for the right side of this, it has to be noted that, since V contains each of the binomials $x-a$, $y-b$, $z-c$, $\ldots$ once and once only, any one of the $1.2.3\ldots n$ terms under Σ will vanish when the substitution

$$x, y, z, \ldots = a, b, c, \ldots$$

is made, unless the denominator of the term also contains *all* the said binomials. But by reason of the interchanges which produce the other denominators, the first term is the only one of this kind: and the value of it after the substitution has been made is

$$(a-b)(a-c) \,.\,.\,.\,.\, (b-a)(b-c) \,.\,.\,.\,.\, (c-a)(c-b) \,.\,.\,.$$

an expression which, as we have already seen in the preceding paper of Jacobi's,* is equal to

$$(-1)^{\frac{1}{2}n(n-1)}P^2.$$

As the left-hand side, kPP', becomes under the same circumstances

$$k \,.\, P^2,$$

we have as our last desideratum

$$k = (-1)^{\frac{1}{2}n(n-1)},$$

and are thus enabled to formulate the proposition

$$\sum\left[\pm\frac{1}{(x-a)(y-b)(z-c)\,.\,.\,.\,.}\right]$$

$$= (-1)^{\frac{1}{2}n(n-1)}\frac{P(x,y,z,\,.\,.\,.).\,P(a,b,c,\,.\,.\,.)}{(x-a)(x-b)(x-c)\,.\,.\,.\,(y-a)(y-b)(y-c)\,.\,.\,.\,(z-a)(z-b)(z-c)\,.\,.\,.},$$

a noteworthy result which in later notation takes the form

$$\begin{vmatrix} (x-a)^{-1} & (x-b)^{-1} & (x-c)^{-1} \,.\,.\,.\,. \\ (y-a)^{-1} & (y-b)^{-1} & (y-c)^{-1} \,.\,.\,.\,. \\ (z-a)^{-1} & (z-b)^{-1} & (z-c)^{-1} \,.\,.\,.\,. \\ \ldots & \ldots & \ldots \end{vmatrix} = (-1)^{\frac{1}{2}n(n-1)}\frac{\zeta^{\frac{1}{2}}(x,y,z,\,.\,.\,.)\,.\,\zeta^{\frac{1}{2}}(a,b,c,\,.\,.\,.)}{F(x)\,.\,F(y)\,.\,F(z)\,.\,.\,.},$$

where n is the number of variables, and

$$F(x) = (x-a)(x-b)(x-c).\,.\,.\,.$$

* Since $V = F(x)\,.\,F(y)\,.\,F(z)\,.\,.\,.\,.$, the first term of the alternating aggregate may be written

$$\frac{F(x)}{x-a}\,.\,\frac{F(y)}{y-b}\,.\,\frac{F(z)}{z-c}\,.\,.\,.\,.$$

which, on the substitution being made, becomes

$$F'(a)\,.\,F'(b)\,.\,F'(c)\,.\,.\,.\,.\,;$$

and it is this form which in Jacobi is replaced by $(-1)^{\frac{1}{2}n(n-1)}P^2$.

On Jacobi's Expansion for the Difference-Product when the Number of Elements is Even. By Thomas Muir, LL.D.

(Read March 19, 1900.)

(1) The character of Jacobi's expansion of this form of alternant will be more readily understood if the two simplest special cases be first considered.

Taking then the case of the 4th order, we have according to Jacobi,

$$\begin{aligned}\zeta^{\frac{1}{2}}(abcd) = (b-a)(d-c)(a^2b^2+c^2d^2) &- (c-a)(d-b)(a^2c^2+b^2d^2)\\ &+ (d-a)(c-b)(a^2d^2+b^2c^2).\end{aligned}$$

No proof is given, but there can be little doubt that he obtained the result by using Laplace's expansion of a determinant as an aggregate of products of complementary minors. Thus

$$\begin{aligned}\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} &= \begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix}\cdot\begin{vmatrix} c^2 & c^3 \\ d^2 & d^3 \end{vmatrix} - \begin{vmatrix} 1 & a \\ 1 & c \end{vmatrix}\cdot\begin{vmatrix} b^2 & b^3 \\ d^2 & d^3 \end{vmatrix} + \begin{vmatrix} 1 & a \\ 1 & d \end{vmatrix}\cdot\begin{vmatrix} b^2 & b^3 \\ c^2 & c^3 \end{vmatrix}\\ &\quad + \begin{vmatrix} 1 & b \\ 1 & c \end{vmatrix}\cdot\begin{vmatrix} a^2 & a^3 \\ d^2 & d^3 \end{vmatrix} - \begin{vmatrix} 1 & b \\ 1 & d \end{vmatrix}\cdot\begin{vmatrix} a^2 & a^3 \\ c^2 & c^3 \end{vmatrix}\\ &\quad + \begin{vmatrix} 1 & c \\ 1 & d \end{vmatrix}\cdot\begin{vmatrix} a^2 & a^2 \\ b^2 & b^3 \end{vmatrix},\end{aligned}$$

and therefore by combining the terms in pairs

$$\zeta^{\frac{1}{2}}(abcd) = \begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix}\cdot\begin{vmatrix} 1 & c \\ 1 & d \end{vmatrix}(c^2d^2+a^2b^2) - \begin{vmatrix} 1 & a \\ 1 & c \end{vmatrix}\cdot\begin{vmatrix} 1 & b \\ 1 & d \end{vmatrix}(b^2d^2+a^2c^2).$$

Applying this process to the next case we have

$$\begin{vmatrix} 1 & a & a^2 & a^3 & a^4 & a^5 \\ 1 & b & b^2 & b^3 & b^4 & b^5 \\ 1 & c & c^2 & c^3 & c^4 & c^5 \\ 1 & d & d^2 & d^3 & d^4 & d^5 \\ 1 & e & e^2 & e^3 & e^4 & e^5 \\ 1 & f & f^2 & f^3 & f^4 & f^5 \end{vmatrix} = \begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix}\cdot|\, c^2d^3e^4f^5 \,| - \begin{vmatrix} 1 & a \\ 1 & c \end{vmatrix}\cdot|\, b^2d^3e^4f^5 \,| + \cdots$$

the number of terms on the right being 15. To each of these, however, when we have removed a monomial factor of the 8th degree, we can employ the preceding case of the theorem, *e.g.*,

$$\begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix} \cdot | c^2d^3e^4f^5 | = \begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix} . c^2d^2e^2f.^2 \; \zeta^{\frac{1}{2}}(cdef),$$

$$\begin{aligned} = (b-a) \cdot c^2d^2e^2f^2. [\, & (d-c)(f-e)(c^2d^2+e^2f^2) \\ & -(e-c)(f-d)(c^2e^2+d^2f^2) \\ & +(f-c)(e-d)(c^2f^2+d^2e^2)], \end{aligned}$$

$$= (b-a)(d-c)(f-e)(c^4d^4e^2f^2 + c^2d^2e^4f^4) - \cdots ;$$

and in this way we shall obtain for $\zeta^{\frac{1}{2}}(abcdef)$ an expression consisting of 45 terms. But when this has been done it will be found that the number can be reduced again to 15 by combining the 45 in a different way into sets of three, viz., by selecting those which have three binomial factors in common. Thus just as the first of the original 15 terms gives rise to the term

$$(b-a)(d-c)(f-e)(c^4d^4e^2f^2 + c^2d^2e^4f^4)$$

the tenth term, $| c^0d^1 | \cdot | a^2b^3e^4f^5 |$, gives rise to

$$(b-a)(d-c)(f-e)(a^4b^4e^2f^2 + d^2b^2e^4f^4),$$

and the fifteenth, $| e^0f^1 | \cdot | a^2b^3c^4d^5 |$,
to

$$(b-a)(d-c)(f-e)(a^4b^4c^2d^2 + a^2b^2c^4d^4);$$

so that one of the resulting 15 terms is

$$(b-a)(d-c)(f-e)(c^4d^4e^2f^2 + c^2d^2e^4f^4 + a^4b^4e^2f^2 + a^2b^2e^4f^4 + a^4b^4c^2d^2 + a^2b^2c^4d^4).$$

Further than this we do not need to go: it is this 15-termed expression which according to Jacobi is the equivalent of $\zeta^{\frac{1}{2}}(abcdef)$.

(2) The two cases may thus be written—

$$\zeta^{\frac{1}{2}}(abcd) = \Sigma(b-a)(d-c).(a^2b^2 + c^2d^2),$$

$$\zeta^{\frac{1}{2}}(abcdef) = \Sigma(b-a)(d-c)(f-e).(a^4b^4c^2d^2 + \ldots\ldots);$$

and the question which naturally arises to the mind of one who looks at them is as to the law of formation of the terms under the symbol of summation and the mode of determining the sign of each.

Jacobi's answer to this is to the effect that he would prefer to

write a_0, a_1, a_2, a_3, a_4, a_5 in place of a, b, c, d, e, f: that having done so and obtained the first term

$$(a_1 - a_0)(a_3 - a_2)(a_5 - a_4)\cdot(a_0{}^4a_1{}^4a_2{}^2a_3{}^2 + a_0{}^2a_1{}^2a_2{}^4a_3{}^4 + \ldots\ldots\ldots),$$

he would then derive two other terms by cyclical permutation of the elements a_3, a_4, a_5; that next he would derive four others from each of these three by cyclical permutation of the elements a_1, a_2, a_3, a_4, a_5; and that the 15 terms got in this way must all be taken positive. His words are :—

"Fingatur expressio

$$(a_1 - a_0)(a_3 - a_2)\ldots\ldots(a_n - a_{n-1})\sum a_2{}^2a_3{}^2a_4{}^4a_5{}^4\ldots\ldots a_{n-1}^{n-1}a_n{}^{n-1},$$

quam quo clarius lex appareat sic scribam,

$$(a_1 - a_0)(a_3 - a_2)\ldots\ldots(a_n - a_{n-1})\sum(a_0a_1)^0(a_2a_3)^2(a_4a_5)^4\ldots\ldots(a_{n-1}a_n)^{n-1},$$

sub signo Σ omni modis permutatis exponentibus,

$$0, 2, 4, \ldots\ldots, n-1.$$

In expressione illa cyclum percurrant *primo* elementa tria,

$$a_{n-2}, a_{n-1}, a_n;$$

Secundo elementa quinque

$$a_{n-4}, a_{n-3}, a_{n-2}, a_{n-1}, a_n;$$

et sic deinceps ita ut *postremo* cyclum percurrant elementa

$$a_1, a_2, a_3, \ldots\ldots, a_n."$$

As has been stated Jacobi confined himself to a mere enunciation of his theorem: in fact, the two Latin sentences just given contain all that he has said in regard to it.

The object of the present paper is to draw attention to a totally different mode of formation of the terms of the expansion, and to establish the accuracy of both modes.

(3) Each term of the expansion, it will have been noted, consists of two parts, (1) a set of linear binomial factors, (2) a single non-linear factor. What we therefore require is a rule for finding the various sets of linear factors, a rule for deriving the single non-linear factor from the set of linear factors to which it is attached, and a rule of signs.

Now to find the various sets of linear factors we have only to

write out in the usual triangular form the $\frac{1}{2}n(n-1)$ differences of the elements, view the differences thus arranged as being the elements of a Pfaffian, and then take the terms of this Pfaffian. For example, in the case of $\zeta^{\frac{1}{2}}(abcd)$ we form the Pfaffian

$$\begin{vmatrix} b-a & c-a & d-a \\ & c-b & d-b \\ & & d-c \end{vmatrix}$$

the expansion of which is

$$(b-a)(d-c) - (c-a)(d-b) + (d-a)(c-b):$$

and this is exactly Jacobi's expansion with the non-linear factors left out. Again, in the case of $\zeta^{\frac{1}{2}}(abcdef)$ we form the Pfaffian

$$\begin{vmatrix} b-a & c-a & d-a & e-a & f-a \\ & c-b & d-b & e-b & f-b \\ & & d-c & e-c & f-c \\ & & & e-d & f-d \\ & & & & f-e \end{vmatrix};$$

take the expansion of it

$$(b-a)(d-c)(f-e) - (b-a)(e-c)(f-d) + (b-a)(e-d)(f-c) - \cdots$$

and all that remains in order to obtain Jacobi's expansion is to annex to each term the appropriate non-linear factor.

No separate rule of signs, it will be observed, is necessary, the signs of the expansion of the auxiliary Pfaffian being exactly the signs of Jacobi's expansion of the difference-product.

(4) Let us now look at the mode of formation of the non-linear factor.

In the case of $\zeta^{\frac{1}{2}}(abcd)$ and the case of $\zeta^{\frac{1}{2}}(abcdef)$ the types are

$$c^2d^2 + a^2b^2,$$
$$c^2d^2e^4f^4 + c^4d^4e^2f^2 + \ldots\ldots;$$

and these, we fortunately observe, resemble determinants, and are, in fact, found to be the permanents

$$\overset{+}{}\begin{vmatrix} 1 & a^2b^2 \\ 1 & c^2d^2 \end{vmatrix}\overset{+}{}, \quad \overset{+}{}\begin{vmatrix} 1 & a^2b^2 & a^4b^4 \\ 1 & c^2d^2 & c^4d^4 \\ 1 & e^2f^2 & e^4f^4 \end{vmatrix}\overset{+}{},$$

so that the complete first term of Jacobi's expansion may be accurately written

$$(b-a)(d-c)\ .\ \overset{+}{|}\,(ab)^0\,(cd)^2\,\overset{+}{|}\,,$$

$$(b-a)(d-c)(f-e)\ .\ \overset{+}{|}\,(ab)^0\,(cd)^2\,(ef)^4\,\overset{+}{|}\,.$$

When, therefore, any one of the sets of linear binomial factors has been obtained, we have only to take the product of the elements in the first factor, the product of the elements in the second factor, and so on : raise these products in order to the 0th, 2nd, 4th, ... powers and form an alternant-like permanent having these powers for the elements of the diagonal.

(5) The theorem in its new form thus is :—

The difference-product of 2n *elements may be expressed as an aggregate of* (2n – 1) (2n – 3) ... 3 . 1 *terms, which are obtainable by taking the ordinary expansion of the Pfaffian whose elements are the* n(2n – 1) *differences arranged in the usual triangular fashion, and then annexing to each term of this expansion an alternant-like permanent whose diagonal elements are the* 0th, 2nd, 4th, ... *powers respectively of the products of the two original elements occurring in each of the linear factors of the term.*

Or, with a freer use of symbols,

The difference-product of $a_1, a_2, a_3, \ldots, a_{2n}$ *is equal to,*

$$\Sigma(a_2-a_1)(a_4-a_3)(a_6-a_5)\ldots(a_{2n}-a_{2n-1})\,.\,\overset{+}{|}\,(a_1a_2)^0\,(a_3a_4)^2\ldots(a_{2n-1}a_{2n})^{2n-2}\,\overset{+}{|}$$

where $(a_2-a_1)(a_4-a_3)\ldots(a_{2n}-{}_{2n-1})$ *is in magnitude and sign a term of*

$$\begin{vmatrix} a_2-a_1 & a_3-a_1 & \ldots & a_{2n}-a_1 \\ & a_3-a_2 & \ldots & a_{2n}-a_2 \\ & & \ldots\ldots & \\ & & & a_{2n}-a_{2n-1} \end{vmatrix},$$

and where each of the n *binary products* $a_1a_2, a_3a_4, \ldots a_{2n-1}a_{2n}$ *is formed from the original elements occurring in one of the linear* n *factors immediately preceding.*

The truth of this may be established as follows:—

From my theorem for the development of a determinant of the $(mn)^{th}$ order * we have, on putting $m=2$, the difference-product of $a_1, a_2, \ldots, a_{2n}$, that is to say, the alternant $|\,a_1{}^0a_2{}^1\ldots a_{2n}^{2n-1}\,|$

* *Trans. Roy. Soc. Edin.*, xxxix. pp. 623–628.

$$= \overset{+}{\Big|} |a_1^0 a_2^1| \, |a_3^2 a_4^3| \, |a_5^4 a_6^5| \ldots |a_{2n-1}^{2n-2} a_{2n}^{2n-1}| \overset{+}{\Big|} - \overset{+}{\Big|} |a_1^0 a_2^1| \, |a_3^2 a_5^3| \, |a_4^4 a_6^5| \ldots |a_{2n-1}^{2n-2} a_{2n}^{2n-1}| \overset{+}{\Big|}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$

or

$$= \sum \overset{+}{\Big|} |a_h^0 a_k^1| \, |a_p^2 a_q^3| \, |a_r^4 a_s^5| \ldots |a_y^{2n-2} a_z^{2n-1}| \overset{+}{\Big|},$$

where $hk, pq, rs, \ldots, yz$ is one of the ways of separating the $2n$ suffixes 1, 2, 3, . . . , $2n$ into n sets of 2 each, and where the sign of summation implies that all other such separations are to be taken, it being understood that the sign preceding any permanent is to be made the same as the sign of that particular term of the alternant which is brought into prominence by the notation employed in specifying the permanent. Since, however, in specifying the typical permanent the particular term of the alternant which makes its appearance is

$$a_h^0 a_k^1 a_p^2 a_q^3 \; \ldots . \; a_y^{2n-2} a_z^{2n-1}$$

the sign of which is $(-1)\mu$ where μ is the number of inverted pairs in $h\,k\,p\,q \ldots y\,z$, a better way of writing this deduction from the general theorem is

$$|a_1^0 a_2^1 \ldots a_{2n}^{2n-1}| = \sum (-)^{\mu} \overset{+}{\Big|} |a_h^0 a_k^1| \, |a_p^2 a_q^3| \, |a_r^4 a_s^5| \ldots |a_y^{2n-2} a_z^{2n-1}| \overset{+}{\Big|},$$

or, more at length,

$$|a_1^0 a_2^1 \ldots a_{2n}^{2n-1}| = \sum (-)^{\mu} \overset{+}{\left|\begin{matrix} |a_h^0 a_k^1| \, |a_h^2 a_k^3| \, |a_h^4 a_k^5| \ldots . |a_h^{2n-2} a_k^{2n-1}| \\ |a_p^0 a_q^1| \, |a_p^2 a_q^3| \, |a_p^4 a_q^5| \ldots . |a_p^{2n-2} a_q^{2n-1}| \\ |a_r^0 a_s^1| \, |a_r^2 a_s^3| \, |a_r^4 a_s^5| \ldots . |a_r^{2n-2} a_s^{2n-1}| \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ |a_y^0 a_z^1| \, |a_y^2 a_z^3| \, |a_y^4 a_z^5| \ldots . |a_y^{2n-2} a_z^{2n-1}| \end{matrix}\right|}^{+} .$$

It is then apparent that on the right the differences $a_k - a_h$, $a_q - a_p$, $a_s - a_r, \ldots, a_z - a_y$ are factors of the 1st, 2nd, 3rd, ... , n^{th} rows respectively, and that if we remove them we shall have

$$|a_1^0 a_2^1 \ldots a_{2n}^{2n-1}| = \sum (-)^{\mu} . (a_k - a_h)(a_q - a_p)(a_s - a_r) \ldots (a_z - a_y)$$

$$\times \overset{+}{\left|\begin{matrix} (a_h a_k)^0 & (a_h a_k)^2 & (a_h a_k)^4 \ldots . & (a_h a_k)^{2n-2} \\ (a_p a_q)^0 & (a_p a_q)^2 & (a_p a_q)^4 \ldots . & (a_p a_q)^{2n-2} \\ (a_r a_s)^0 & (a_r a_s)^2 & (a_r a_s)^4 \ldots . & (a_r a_s)^{2n-2} \\ \ldots & \ldots & \ldots & \ldots \\ (a_y a_z)^0 & (a_y a_z)^2 & (a_y a_z)^4 \ldots . & (a_y a_z)^{2n-2} \end{matrix}\right|}^{+} .$$

But to say that hk, pq, rs, . . . , yz is a grouping of 1, 2, 3, . . . , $2n$ into pairs, and that μ is the number of inverted pairs in $h\ k\ p\ q\ r\ s \ldots y\ z$ is exactly the same as to say that

$$(-)^{\mu}.(a_k - a_h)\ (a_q - a_p)\ (a_s - a_r)\ .\ .\ .\ (a_z - a_y)$$

is a term of the Pfaffian

$$\left|\begin{array}{cccc} a_2 - a_1 & a_3 - a_1 & \cdots\cdots & a_{2n} - a_1 \\ & a_3 - a_2 & \cdots\cdots & a_{2n} - a_2 \\ & & \cdots\cdots & \cdots \\ & & & a_{2n} - a_{2n-1} \end{array}\right|,$$

where, be it observed, the suffixes of the two a's in any element, are, when reversed, the place-numbers of the element.

The theorem is thus fully established.

(6) It is worth noting that in the general theorem used at the outset of the preceding demonstration the number of terms in the development is

$$\frac{(mn)!}{m^n.\ (n!)};$$

and that this in the case where $m = 2$ becomes

$$\frac{1\ .\ 3\ .\ 5\ .\ 7\ .\ .\ .\ .\ (2n-1)\ .\ 2\ .\ 4\ .\ 6\ .\ .\ .\ .\ 2n}{2n\ .\ 1\ .\ 2\ .\ 3\ .\ .\ .\ .\ n}$$

or

$$1\ .\ 3\ .\ 5\ .\ 7\ .\ .\ .\ .\ (2n-1),$$

which is, as was to be expected, the well-known expression for the number of terms in a Pfaffian of the n^{th} order.

Of course, in writing the various grouping of 1, 2, 3, . . . , $2n$ into pairs it is desirable to write the members of each pair in ascending order, and also to have all the first members of the pairs in ascending order.

(7) On account of the co-existence of two rules for obtaining the same development, one of which is the rule for the expansion of a Pfaffian, it follows that the other rule, Jacobi's, must contain within it the substance of the Pfaffian definition.

This implied definition may be formulated as follows:—

One term of the Pfaffian—

$$\left|\begin{array}{ccccc} 12 & 13 & 14 & \ldots\ldots & 1,2n \\ & 23 & 24 & \ldots\ldots & 2,2n \\ & & 34 & \ldots\ldots & 3,2n \\ & & & \ldots\ldots\ldots & \\ & & & & 2n-1,2n \end{array}\right|$$

is

$$12\,.\,34\,.\,56\,\ldots\ldots(2n-1,\,2n)\,;$$

this is increased to three (1×3) *by performing upon it the cyclical substitution*

$$\begin{pmatrix} 2n-2, & 2n-1, & 2n \\ 2n-1, & 2n, & 2n-2 \end{pmatrix};$$

these three are increased to fifteen $(1\cdot3\cdot5)$ *by performing on each the cyclical substitution*

$$\begin{pmatrix} 2n-4, & 2n-3, & 2n-2, & 2n-1, & 2n \\ 2n-3, & 2n-2, & 2n-1, & 2n, & 2n-4 \end{pmatrix};$$

and so on: all the terms being initially positive, but the sign of any one being changed as often as it is necessary to put the members of an inverted pair into their natural order.

Thus the first term of the Pfaffian of the 3rd order

$$\left|\begin{array}{ccccc} 12 & 13 & 14 & 15 & 16 \\ & 23 & 24 & 25 & 26 \\ & & 34 & 35 & 36 \\ & & & 45 & 46 \\ & & & & 56 \end{array}\right|$$

is $12\cdot34\cdot56$; and by the cyclical permutation of 456 we obtain two others

$$+\ 12\cdot35\cdot64\ +\ 12\cdot36\cdot45$$

or

$$-\ 12\cdot35\cdot46\ +\ 12\cdot36\cdot45\,;$$

and lastly from these three by the cyclical permutation of 23456 we obtain the remaining twelve terms.

No other definition shows so clearly that the total number of terms in a Pfaffian of the n^{th} order is $1\cdot3\cdot5\cdot7\ldots(2n-1)$.

(8) A similar definition of a determinant is at once suggested, *viz.*,

One term of the determinant $|a_1b_2c_3 \ldots z_n|$ *is* $+a_1b_2c_3 \ldots z_n$: *this is increased to two by the cyclical permutation of* $n-1, n$ *accompanied by change of sign: these two are increased to six (i.e.* 2×3) *by the cyclical permutation of* $n-2, n-1, n$ *without alteration of sign: then these six are increased to twenty-four (i.e.* $2 \times 3 \times 4$, *by the cyclical permutation of* $n-3, n-2, n-1, n$ *accompanied by change of sign: and so on.*

Thus, the first term of $|a_1b_2c_3d_4|$ is

$$+a_1b_2c_3d_4,$$

from which by cyclical permutation of 3, 4 we obtain another

$$-a_1b_2c_4d_3;$$

then by cyclical permutation of 234 without change of sign we derive from the former

$$+\ a_1b_3c_4d_2\ +\ a_1b_4c_2d_3,$$

and from the latter

$$-\ a_1b_3c_2d_4\ -\ a_1b_4c_3d_2;$$

and lastly by cyclical permutation of 1234 and change of sign there is derived from these six the remaining eighteen:

As before, the total number of terms, viz., $1 \cdot 2 \cdot 3 \ldots n$, is brought very clearly into evidence.

On certain Aggregates of Determinant Minors.

By **Thomas Muir**, LL.D.

(Read March 5, 1900.)

(1) Two curious identities have been established regarding certain aggregates of minors of special determinants; the first, which concerns *axisymmetric* determinants, having been discovered by Kronecker in 1882,* and the second, which concerns *centro-symmetric* determinants, having been published by me in 1888.† When we come to think of the possibility of generalising these identities, it is readily seen that there are at least two lines of attack which suggest themselves on reading the mere description of the kind of identity; for, in saying that the identities deal with "an aggregate of minor determinants of a special determinant," we are conscious of two points of limitation in the description, the one signalised by the word "minor" and the other by the word "special." If, therefore, an identity were obtained regarding an aggregate of which the terms were determinants unrestricted by a family relationship, we might have one form of generalisation; and if, while retaining the family relationship, we succeeded in removing the restriction as to the form of the parent, a generalisation of a different type might be the result.

The former of these lines of attack I have followed up on a previous occasion; in the present paper I take the latter line.

(2) Kronecker's theorem, it will be remembered, is to the effect that the aggregates

$$\begin{vmatrix} 1\,2 \\ 3\,4 \end{vmatrix} - \begin{vmatrix} 1\,3 \\ 2\,4 \end{vmatrix} + \begin{vmatrix} 1\,4 \\ 2\,3 \end{vmatrix},$$

$$\begin{vmatrix} 1\,2\,3 \\ 4\,5\,6 \end{vmatrix} - \begin{vmatrix} 1\,2\,4 \\ 3\,5\,6 \end{vmatrix} + \begin{vmatrix} 1\,2\,5 \\ 3\,4\,6 \end{vmatrix} - \begin{vmatrix} 1\,2\,6 \\ 3\,4\,5 \end{vmatrix},$$

$$\begin{vmatrix} 1\,2\,3\,4 \\ 5\,6\,7\,8 \end{vmatrix} - \begin{vmatrix} 1\,2\,3\,5 \\ 4\,6\,7\,8 \end{vmatrix} + \begin{vmatrix} 1\,2\,3\,6 \\ 4\,5\,7\,8 \end{vmatrix} - \begin{vmatrix} 1\,2\,3\,7 \\ 4\,5\,6\,8 \end{vmatrix} + \begin{vmatrix} 1\,2\,3\,8 \\ 4\,5\,6\,7 \end{vmatrix},$$

. . . .

* Kronecker, L., "Die Subdeterminanten symmetrischer Systeme," *Sitzungsb. d. k. Akad. d. Wiss.*, 1882, pp. 821–824.

† Muir, T., "On Vanishing Aggregates of Determinants," *Proc. Roy. Soc. Edin.*, xv. pp. 96–105.

vanish in the case of axisymmetric determinants of the 4th, 6th, 8th, . . . orders respectively. Removing, then, the restriction as to the form of the parent determinant, $\begin{vmatrix}1&2&3&4&5&6\\1&2&3&4&5&6\end{vmatrix}$ say, and expanding each of the specified minors in terms of the elements of the last row and their cofactors, we have

$$\begin{vmatrix}1\,2\,3\\4\,5\,6\end{vmatrix}-\begin{vmatrix}1\,2\,4\\3\,5\,6\end{vmatrix}+\begin{vmatrix}1\,2\,5\\3\,4\,6\end{vmatrix}-\begin{vmatrix}1\,2\,6\\3\,4\,5\end{vmatrix}$$

$$\begin{aligned}=\begin{vmatrix}1\,2\\4\,5\end{vmatrix}\cdot\begin{matrix}3\\6\end{matrix}-\begin{vmatrix}1\,2\\4\,6\end{vmatrix}\cdot\begin{matrix}3\\5\end{matrix}+\begin{vmatrix}1\,2\\5\,6\end{vmatrix}\cdot\begin{matrix}3\\4\end{matrix}\\
-\begin{vmatrix}1\,2\\3\,5\end{vmatrix}\cdot\begin{matrix}4\\6\end{matrix}+\begin{vmatrix}1\,2\\3\,6\end{vmatrix}\cdot\begin{matrix}4\\5\end{matrix}-\begin{vmatrix}1\,2\\5\,6\end{vmatrix}\cdot\begin{matrix}4\\3\end{matrix}\\
+\begin{vmatrix}1\,2\\3\,4\end{vmatrix}\cdot\begin{matrix}5\\6\end{matrix}-\begin{vmatrix}1\,2\\3\,6\end{vmatrix}\cdot\begin{matrix}5\\4\end{matrix}+\begin{vmatrix}1\,2\\4\,6\end{vmatrix}\cdot\begin{matrix}5\\3\end{matrix}\\
-\begin{vmatrix}1\,2\\3\,4\end{vmatrix}\cdot\begin{matrix}6\\5\end{matrix}+\begin{vmatrix}1\,2\\3\,5\end{vmatrix}\cdot\begin{matrix}6\\4\end{matrix}-\begin{vmatrix}1\,2\\4\,5\end{vmatrix}\cdot\begin{matrix}6\\3\end{matrix}.\end{aligned}$$

Now the twelve minors of the second order which occur here are not all different, the real state of matters being that we have two appearances of each of the six minors formable from the two curtailed rows

$$\begin{matrix}1&1&1&1\\3,&4,&5,&6,\end{matrix}$$

$$\begin{matrix}2&2&2&2\\3,&4,&5,&6.\end{matrix}$$

Taking advantage of this we find our aggregate equal to

$$\begin{aligned}\begin{vmatrix}1\,2\\4\,5\end{vmatrix}\left(\begin{matrix}3\\6\end{matrix}-\begin{matrix}6\\3\end{matrix}\right)-\begin{vmatrix}1\,2\\4\,6\end{vmatrix}\left(\begin{matrix}3\\5\end{matrix}-\begin{matrix}5\\3\end{matrix}\right)+\begin{vmatrix}1\,2\\5\,6\end{vmatrix}\left(\begin{matrix}3\\4\end{matrix}-\begin{matrix}4\\3\end{matrix}\right)\\
-\begin{vmatrix}1\,2\\3\,5\end{vmatrix}\left(\begin{matrix}4\\6\end{matrix}-\begin{matrix}6\\4\end{matrix}\right)+\begin{vmatrix}1\,2\\3\,6\end{vmatrix}\left(\begin{matrix}4\\5\end{matrix}-\begin{matrix}5\\4\end{matrix}\right)\\
+\begin{vmatrix}1\,2\\3\,4\end{vmatrix}\left(\begin{matrix}5\\6\end{matrix}-\begin{matrix}6\\5\end{matrix}\right),\end{aligned}$$

where the cofactor of every two-lined minor is the difference between a pair of conjugate elements of the parent determinant.

The fact that axisymmetry implies equality of conjugate elements thus accounts at once for Kronecker's theorem.

(3) Proceeding in an exactly similar way we change

$$\begin{vmatrix}1234\\5678\end{vmatrix} - \begin{vmatrix}1235\\4678\end{vmatrix} + \begin{vmatrix}1236\\4578\end{vmatrix} - \begin{vmatrix}1237\\4568\end{vmatrix} + \begin{vmatrix}1238\\4567\end{vmatrix}$$

into

$$\begin{aligned}
&\begin{vmatrix}123\\567\end{vmatrix}\cdot\begin{matrix}4\\8\end{matrix} - \begin{vmatrix}123\\568\end{vmatrix}\cdot\begin{matrix}4\\7\end{matrix} + \begin{vmatrix}123\\578\end{vmatrix}\cdot\begin{matrix}4\\6\end{matrix} - \begin{vmatrix}123\\678\end{vmatrix}\cdot\begin{matrix}4\\5\end{matrix}\\
&\qquad - \begin{vmatrix}123\\467\end{vmatrix}\cdot\begin{matrix}5\\8\end{matrix} + \begin{vmatrix}123\\468\end{vmatrix}\cdot\begin{matrix}5\\7\end{matrix} - \begin{vmatrix}123\\478\end{vmatrix}\cdot\begin{matrix}5\\6\end{matrix} + \begin{vmatrix}123\\678\end{vmatrix}\cdot\begin{matrix}5\\4\end{matrix}\\
&\qquad\qquad + \begin{vmatrix}123\\457\end{vmatrix}\cdot\begin{matrix}6\\8\end{matrix} - \begin{vmatrix}123\\458\end{vmatrix}\cdot\begin{matrix}6\\7\end{matrix} + \begin{vmatrix}123\\478\end{vmatrix}\cdot\begin{matrix}6\\5\end{matrix} - \cdots\\
&\qquad\qquad\qquad - \begin{vmatrix}123\\456\end{vmatrix}\cdot\begin{matrix}7\\8\end{matrix} + \begin{vmatrix}123\\458\end{vmatrix}\cdot\begin{matrix}7\\6\end{matrix} - \cdots\\
&\qquad\qquad\qquad\qquad + \begin{vmatrix}123\\456\end{vmatrix}\cdot\begin{matrix}8\\7\end{matrix} - \cdots
\end{aligned}$$

where we have now 4×5 terms, each of which is the product of a three-lined determinant and a simple element. On examination it will be seen that the simple elements consist of the ten

$$\left.\begin{array}{rrrr}
4 & 4 & 4 & 4\\
8, & 7, & 6, & 5\\
 & 5 & 5 & 5\\
 & 8, & 7, & 6\\
 & & 6 & 6\\
 & & 8, & 7\\
 & & & 7\\
 & & & 8
\end{array}\right\}\text{ and their conjugates }\left\{\begin{array}{rrrr}
8 & 7 & 6 & 5\\
4, & 4, & 4, & 4\\
 & 8 & 7 & 6\\
 & 5, & 5, & 5\\
 & & 8 & 7\\
 & & 6, & 6\\
 & & & 8\\
 & & & 7,
\end{array}\right.$$

and that the twenty corresponding three-lined determinants consist of the ten

$$
\begin{array}{cccc}
\begin{vmatrix}1\,2\,3\\5\,6\,7\end{vmatrix} & \begin{vmatrix}1\,2\,3\\5\,6\,8\end{vmatrix} & \begin{vmatrix}1\,2\,3\\5\,7\,8\end{vmatrix} & \begin{vmatrix}1\,2\,3\\6\,7\,8\end{vmatrix} \\
 & \begin{vmatrix}1\,2\,3\\4\,6\,7\end{vmatrix} & \begin{vmatrix}1\,2\,3\\4\,6\,8\end{vmatrix} & \begin{vmatrix}1\,2\,3\\4\,7\,8\end{vmatrix} \\
 & & \begin{vmatrix}1\,2\,3\\4\,5\,7\end{vmatrix} & \begin{vmatrix}1\,2\,3\\4\,5\,8\end{vmatrix} \\
 & & & \begin{vmatrix}1\,2\,3\\4\,5\,6\end{vmatrix}
\end{array}
$$

used twice over, the accompanying sign being changed in the case of the second occurrence. The resulting expansion thus is

$$
\begin{aligned}
&\begin{vmatrix}1\,2\,3\\5\,6\,7\end{vmatrix}\begin{pmatrix}4 & - & 8\\8 & & 4\end{pmatrix} - \begin{vmatrix}1\,2\,3\\5\,6\,8\end{vmatrix}\begin{pmatrix}4 & - & 7\\7 & & 4\end{pmatrix} + \begin{vmatrix}1\,2\,3\\5\,7\,8\end{vmatrix}\begin{pmatrix}4 & - & 6\\6 & & 4\end{pmatrix} - \begin{vmatrix}1\,2\,3\\6\,7\,8\end{vmatrix}\begin{pmatrix}4 & - & 5\\5 & & 4\end{pmatrix} \\
&\qquad - \begin{vmatrix}1\,2\,3\\4\,6\,7\end{vmatrix}\begin{pmatrix}5 & - & 8\\8 & & 5\end{pmatrix} + \begin{vmatrix}1\,2\,3\\4\,6\,8\end{vmatrix}\begin{pmatrix}5 & - & 7\\7 & & 5\end{pmatrix} - \begin{vmatrix}1\,2\,3\\4\,7\,8\end{vmatrix}\begin{pmatrix}5 & - & 6\\6 & & 5\end{pmatrix} \\
&\qquad\qquad + \begin{vmatrix}1\,2\,3\\4\,5\,7\end{vmatrix}\begin{pmatrix}6 & - & 8\\8 & & 6\end{pmatrix} - \begin{vmatrix}1\,2\,3\\4\,5\,8\end{vmatrix}\begin{pmatrix}6 & - & 7\\7 & & 6\end{pmatrix} \\
&\qquad\qquad\qquad - \begin{vmatrix}1\,2\,3\\4\,5\,6\end{vmatrix}\begin{pmatrix}7 & - & 8\\8 & & 7\end{pmatrix}.
\end{aligned}
$$

(4) The general theorem to which we are led is, of course, readily enunciated when the law of formation of the terms on the right-hand side of the identity is grasped. This is most easily done by first considering the two triangular arrangements of elements which go to form the second factors of the terms. The parent determinant being of the $(2n)^{\text{th}}$ order, the first of these arrangements is

$$
\begin{array}{ccccc}
\begin{matrix}n\\2n\end{matrix} & \begin{matrix}n\\2n-1\end{matrix} & \begin{matrix}n\\2n-2\end{matrix} & \cdots\cdots & \begin{matrix}n\\n+1\end{matrix} \\
 & \begin{matrix}n+1\\2n\end{matrix} & \begin{matrix}n+1\\2n-1\end{matrix} & \cdots\cdots & \begin{matrix}n+1\\n+2\end{matrix} \\
 & & \begin{matrix}n+2\\2n\end{matrix} & \cdots\cdots & \begin{matrix}n+2\\n+3\end{matrix} \\
 & & & \cdots\cdots\cdots & \\
 & & & & \begin{matrix}2n-1\\2n\end{matrix},
\end{array}
$$

and the other is got from it by substituting for each element its conjugate. The full set of binomial factors is thus

$$\frac{n}{2n} - \frac{2n}{n}, \quad \frac{n}{2n-1} - \frac{2n-1}{n}, \quad \ldots .$$

$$\frac{n+1}{2n} - \frac{2n}{n+1}, \quad \ldots .$$

$$\ldots .$$

As for the determinant which is to be the cofactor of any of these differences, its two lines of indices must contain exactly all the indices not found in the said difference, the first line being always

$$1, 2, 3, \ldots, n-1,$$

and the second being therefore got from

$$n, n+1, n+2, \ldots, 2n$$

by dropping out the two indices which appear in the annexed difference. This being grasped, there then only remains to be determined the law of signs of the terms. Looking again to the triangle of elements we at once observe that in each of the left-to-right lines of the triangle the signs are alternately positive and negative, and that so also are the first signs of the various rows taken in order. If in addition to this we only note the fact that as we thus move from place to place in the triangle, there is a corresponding alteration in the sum of the indices of the binomial factors, we see that the determination of the sign of any term can be made dependent on the difference between the sum of the indices of its binomial factor and the sum of the indices of the first binomial factor of all.

Our enunciation of the general theorem will thus take the following form:—

If μ and ν be any integers, μ being the less, taken from the series

$$n, n+1, n+2, \ldots, 2n;$$

and $\alpha, \beta, \gamma, \ldots, \omega$ be what the series becomes when μ is removed, and $\alpha, \beta, \gamma, \ldots, \psi$ what it becomes when both are removed; then in connection with any even-ordered determinant $\begin{vmatrix} 1 & 2 & 3 & \ldots & 2n \\ 1 & 2 & 3 & \ldots & 2n \end{vmatrix}$

we have

$$\sum_{\mu}(-)^{n-\mu}\begin{vmatrix}1 & 2 & 3 \ldots n-1, \mu \\ \alpha & \beta & \gamma \ldots\ldots, \omega\end{vmatrix} = \sum_{\mu,\nu}(-)^{3n-(\mu+\nu)}\begin{vmatrix}1\,2\,3\ldots n-1 \\ \alpha\,\beta\,\gamma\ldots\ldots\,\psi\end{vmatrix}\left(\frac{\mu}{\nu}-\frac{\nu}{\mu}\right).$$

(5) From this, of course, a variant enunciation of Kronecker's theorem at once follows, viz.,

If μ *be any integer taken from the series* $n, n+1, n+2, \ldots, 2n$ *and* $\alpha, \beta, \gamma, \ldots, \omega$ *be what the series then becomes, then in connection with any even-ordered determinant* $\begin{vmatrix}1 & 2 & 3 & \ldots & 2n \\ 1 & 2 & 3 & \ldots & 2n\end{vmatrix}$ *whose coaxial minor* $\begin{vmatrix}n, n+1, n+2, \ldots, 2n \\ n, n+1, n+2, \ldots, 2n\end{vmatrix}$ *is axisymmetric, we have*

$$\sum_{\mu}(-)^{n-\mu}\begin{vmatrix}1 & 2 & 3 & \ldots & n-1, \mu \\ \alpha & \beta & \gamma & \ldots\ldots & \omega\end{vmatrix} = 0.$$

The advantage of this form of enunciation lies in the fact that it localises the axisymmetry which is necessary for the validity of *one* of Kronecker's identities, and thus by implication indicates the number of such identities which hold in the case where the axisymmetry of the parent determinant is complete. This number is clearly the number of coaxial minors of the $(n+1)^{\text{th}}$ order contained in a determinant of the $(2n)^{\text{th}}$ order, *i.e.*, $C_{2n,\,n-1}$. The same, of course, is also evident from the fact that instead of taking $1, 2, 3, \ldots, n-1$ for constant indices in the first line, we might with equal reason select any other $n-1$ indices from the $2n$ available.

(6) With the general theorem now in our possession, other special cases of it similar to Kronecker's can easily be obtained. Perhaps the most important of these is that where the coaxial minor of the parent determinant is *skew*. To get this we have only to substitute $-\frac{\mu}{\nu}$ for $\frac{\nu}{\mu}$ in the general enunciation, the result being:—

In connection with any even-ordered determinant $\begin{vmatrix}1 & 2 & 3 & \ldots & 2n \\ 1 & 2 & 3 & \ldots & 2n\end{vmatrix}$ *whose coaxial minor* $\begin{vmatrix}n, n+1, n+2, \ldots, 2n \\ n, n+1, n+2, \ldots, 2n\end{vmatrix}$ *is skew, we have*

$$\sum_{\mu}(-)^{n-\mu}\begin{vmatrix}1 & 2 & 3 \ldots n-1, n \\ \alpha & \beta & \gamma \ldots\ldots, \omega\end{vmatrix} = 2\sum_{\mu,\nu}(-)^{3n-(\mu+\nu)}\begin{vmatrix}1\,2\,3\ldots n-1 \\ \alpha\,\beta\,\gamma\ldots\ldots\,\psi\end{vmatrix}\cdot\frac{\mu}{\nu}.$$

(7) Both the general identity, however, and the special cases acquire new significance if we make use of a recently discovered theorem regarding Pfaffians in order to alter the form of the right-hand side of the identity.

This theorem, in so far as it concerns the present subject, may be described as giving an expansion of a special Pfaffian in the form of a series of terms, each of which is the product of a determinant and an element of the Pfaffian, the specialty of the Pfaffian being that the elements in the places where $n-1$ of the $2n$ frame-lines intersect are zeros.

Thus the Pfaffian

$$\left|\begin{array}{ccccc} \cdot & a_3 & a_4 & a_5 & a_6 \\ & b_3 & b_4 & b_5 & b_6 \\ & & c_4 & c_5 & c_6 \\ & & & d_5 & d_6 \\ & & & & e_6 \end{array}\right|,$$

which is of the 3rd order, and has a zero at the place (12) where two of the frame-lines intersect, is equal to

$$-|a_3b_4|e_6 + |a_3b_5|d_6 - |a_3b_6|d_5 - |a_4b_5|c_6 + \ a_4b_6|c_5 - |a_5b_6|c_4,$$

where the first factors of the terms are the six determinants formed from

$$\begin{array}{cccc} a_3 & a_4 & a_5 & a_6 \\ b_3 & b_4 & b_5 & b_6 \end{array}$$

and the second factors are the remaining non-zero elements

$$\begin{array}{ccc} c_4 & c_5 & c_6 \\ & d_5 & d_6 \\ & & e_6 . \end{array}$$

Similarly the Pfaffian

$$\left|\begin{array}{ccccccc} \cdot & \cdot & a_4 & a_5 & a_6 & a_7 & a_8 \\ & \cdot & b_4 & b_5 & b_6 & b_7 & b_8 \\ & & c_4 & c_5 & c_6 & c_7 & c_8 \\ & & & d_5 & d_6 & d_7 & d_8 \\ & & & & e_6 & e_7 & e_8 \\ & & & & & f_7 & f_8 \\ & & & & & & g_8 \end{array}\right|,$$

which is of the 4th order, and has a zero at the places (12), (13), (23), where three of the frame-lines intersect, is equal to

$$|a_4b_5c_6|\cdot g_8 - |a_4b_5c_7|\cdot f_8 + |a_4b_5c_8|\cdot f_7 + |a_4b_6c_7|\cdot e_8$$
$$- |a_4b_6c_8|\cdot e_7 + |a_4b_7c_8|\cdot e_6 - |a_5b_6c_7|\cdot d_8 + |a_5b_6c_8|\cdot d_7$$
$$- |a_5b_7c_8|\cdot d_6 + |a_6b_7c_8|\cdot d_5\,,$$

where the first factors of the ten terms of the expansion are the three-lined determinants formable from the rectangular array

$$\begin{matrix} a_4 & a_5 & a_6 & a_7 & a_8 \\ b_4 & b_5 & b_6 & b_7 & b_8 \\ c_4 & c_5 & c_6 & c_7 & c_8 \end{matrix}$$

and the second factors are the remaining non-zero elements

$$\begin{matrix} d_5 & d_6 & d_7 & d_8 \\ & e_6 & e_7 & e_8 \\ & & f_7 & f_8 \\ & & & g_8\,. \end{matrix}$$

(8) Now, on referring back, it will be found that this kind of expansion is exactly similar to that which appears on the right-hand side of the new general identity of § 4. This latter in the special case of § 3 may consequently be written—

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 3 & 5 \\ 4 & 6 & 7 & 8 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 & 6 \\ 4 & 5 & 7 & 8 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 3 & 7 \\ 4 & 5 & 6 & 8 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 & 8 \\ 4 & 5 & 6 & 7 \end{vmatrix}$$

$$= -\left|\begin{matrix} \cdot & \cdot & \begin{smallmatrix}1\\4\end{smallmatrix} & \begin{smallmatrix}1\\5\end{smallmatrix} & \begin{smallmatrix}1\\6\end{smallmatrix} & \begin{smallmatrix}1\\7\end{smallmatrix} & \begin{smallmatrix}1\\8\end{smallmatrix} \\[2ex] & \cdot & \begin{smallmatrix}2\\4\end{smallmatrix} & \begin{smallmatrix}2\\5\end{smallmatrix} & \begin{smallmatrix}2\\6\end{smallmatrix} & \begin{smallmatrix}2\\7\end{smallmatrix} & \begin{smallmatrix}2\\8\end{smallmatrix} \\[2ex] & & \begin{smallmatrix}3\\4\end{smallmatrix} & \begin{smallmatrix}3\\5\end{smallmatrix} & \begin{smallmatrix}3\\6\end{smallmatrix} & \begin{smallmatrix}3\\7\end{smallmatrix} & \begin{smallmatrix}3\\8\end{smallmatrix} \\[2ex] & & & \begin{smallmatrix}4&&5\\&-&\\5&&4\end{smallmatrix} & \begin{smallmatrix}4&&6\\&-&\\6&&4\end{smallmatrix} & \begin{smallmatrix}4&&7\\&-&\\7&&4\end{smallmatrix} & \begin{smallmatrix}4&&8\\&-&\\8&&4\end{smallmatrix} \\[2ex] & & & & \begin{smallmatrix}5&&6\\&-&\\6&&5\end{smallmatrix} & \begin{smallmatrix}5&&7\\&-&\\7&&5\end{smallmatrix} & \begin{smallmatrix}5&&8\\&-&\\8&&5\end{smallmatrix} \\[2ex] & & & & & \begin{smallmatrix}6&&7\\&-&\\7&&6\end{smallmatrix} & \begin{smallmatrix}6&&8\\&-&\\8&&6\end{smallmatrix} \\[2ex] & & & & & & \begin{smallmatrix}7&&8\\&-&\\8&&7\end{smallmatrix} \end{matrix}\right|,$$

the second side of which, as before, manifestly vanishes when the parent determinant is axisymmetric, and becomes

$$
2\left|\begin{array}{ccccccc}
\cdot & \cdot & \substack{1\\4} & \substack{1\\5} & \substack{1\\6} & \substack{1\\7} & \substack{1\\8} \\
 & & \substack{2\\4} & \substack{2\\5} & \substack{2\\6} & \substack{2\\7} & \substack{2\\8} \\
 & & \substack{3\\4} & \substack{3\\5} & \substack{3\\6} & \substack{3\\7} & \substack{3\\8} \\
 & & & \substack{4\\5} & \substack{4\\6} & \substack{4\\7} & \substack{4\\8} \\
 & & & & \substack{5\\6} & \substack{5\\7} & \substack{5\\8} \\
 & & & & & \substack{6\\7} & \substack{6\\8} \\
 & & & & & & \substack{7\\8}
\end{array}\right|
$$

when the parent determinant is skew.

(9) Leaving now this subject—which has been led up to by a consideration of Kronecker's theorem—let us turn to a similar inquiry connected with my analogous theorem of 1888.

The latter is to the effect that the aggregates

$$
\begin{vmatrix}31\\34\end{vmatrix} - \begin{vmatrix}34\\31\end{vmatrix} + \begin{vmatrix}24\\34\end{vmatrix} - \begin{vmatrix}34\\24\end{vmatrix},
$$

$$
\begin{vmatrix}451\\456\end{vmatrix} - \begin{vmatrix}456\\451\end{vmatrix} + \begin{vmatrix}426\\456\end{vmatrix} - \begin{vmatrix}456\\426\end{vmatrix} + \begin{vmatrix}356\\456\end{vmatrix} - \begin{vmatrix}456\\356\end{vmatrix},
$$

$$
\left.\begin{array}{l}
\begin{vmatrix}5671\\5678\end{vmatrix} + \begin{vmatrix}5628\\5678\end{vmatrix} + \begin{vmatrix}5378\\5678\end{vmatrix} + \begin{vmatrix}4678\\5678\end{vmatrix} \\
-\begin{vmatrix}5678\\5671\end{vmatrix} - \begin{vmatrix}5678\\5628\end{vmatrix} - \begin{vmatrix}5678\\5378\end{vmatrix} - \begin{vmatrix}5678\\4678\end{vmatrix}
\end{array}\right\},
$$

. . . .

vanish in the case of centro-symmetric determinants of the 4^{th}, 6^{th} 8^{th}, ..., orders respectively.

From this, as before, we remove the restriction as to the form of the parent determinant, $\begin{vmatrix}123456\\123456\end{vmatrix}$ say, and expand each minor of the aggregate in terms of three elements and their cofactors, the three elements in the six cases being those of the 3rd row 3rd column, 2nd row 2nd column, 1st row 1st column respectively. The result of this is

$$\begin{vmatrix}451\\456\end{vmatrix} - \begin{vmatrix}456\\451\end{vmatrix} + \begin{vmatrix}426\\456\end{vmatrix} - \begin{vmatrix}456\\426\end{vmatrix} + \begin{vmatrix}356\\456\end{vmatrix} - \begin{vmatrix}456\\356\end{vmatrix}$$

$$= \begin{vmatrix}45\\45\end{vmatrix}\cdot\begin{matrix}1\\6\end{matrix} - \begin{vmatrix}45\\46\end{vmatrix}\cdot\begin{matrix}1\\5\end{matrix} + \begin{vmatrix}45\\56\end{vmatrix}\cdot\begin{matrix}1\\4\end{matrix} - \begin{vmatrix}45\\45\end{vmatrix}\cdot\begin{matrix}6\\1\end{matrix} + \begin{vmatrix}46\\45\end{vmatrix}\cdot\begin{matrix}5\\1\end{matrix} - \begin{vmatrix}56\\45\end{vmatrix}\cdot\begin{matrix}4\\1\end{matrix}$$

$$- \begin{vmatrix}46\\45\end{vmatrix}\cdot\begin{matrix}2\\6\end{matrix} + \begin{vmatrix}46\\46\end{vmatrix}\cdot\begin{matrix}2\\5\end{matrix} - \begin{vmatrix}46\\56\end{vmatrix}\cdot\begin{matrix}2\\4\end{matrix} + \begin{vmatrix}45\\46\end{vmatrix}\cdot\begin{matrix}6\\2\end{matrix} - \begin{vmatrix}46\\46\end{vmatrix}\cdot\begin{matrix}5\\2\end{matrix} + \begin{vmatrix}56\\46\end{vmatrix}\cdot\begin{matrix}4\\2\end{matrix}$$

$$+ \begin{vmatrix}56\\45\end{vmatrix}\cdot\begin{matrix}3\\6\end{matrix} - \begin{vmatrix}56\\46\end{vmatrix}\cdot\begin{matrix}3\\5\end{matrix} + \begin{vmatrix}56\\56\end{vmatrix}\cdot\begin{matrix}3\\4\end{matrix} - \begin{vmatrix}45\\56\end{vmatrix}\cdot\begin{matrix}6\\3\end{matrix} + \begin{vmatrix}46\\56\end{vmatrix}\cdot\begin{matrix}5\\3\end{matrix} - \begin{vmatrix}56\\56\end{vmatrix}\cdot\begin{matrix}4\\3\end{matrix},$$

where, it is worthy of notice, each of the three lines on the right-hand side of the identity contains the expansion of two minors which are conjugate to one another, this arrangement being made for the purpose of showing more clearly that the eighteen two-lined minors which appear in the expansion, consist merely of the nine such minors formable from $\begin{vmatrix}456\\456\end{vmatrix}$, and that each of these nine occurs first with one of the elements of $\begin{vmatrix}123\\456\end{vmatrix}$ as a cofactor, and then with one of the elements of $\begin{vmatrix}456\\123\end{vmatrix}$. This suffices to draw attention in passing to the fact, which can also be reached by consideration of the left-hand side, that the identity involves all the elements of $\begin{vmatrix}123456\\123456\end{vmatrix}$ except those of the minor $\begin{vmatrix}123\\123\end{vmatrix}$, and that each element of $\begin{vmatrix}456\\456\end{vmatrix}$ occurs four times, while each element of $\begin{vmatrix}123\\456\end{vmatrix}$ and its conjugate $\begin{vmatrix}456\\123\end{vmatrix}$ occurs only once. It is thus seen that the right-hand side may be condensed into

$$\begin{vmatrix}4\,5\\4\,5\end{vmatrix}\left(\begin{matrix}1\\6\end{matrix}-\begin{matrix}6\\1\end{matrix}\right)-\begin{vmatrix}4\,5\\4\,6\end{vmatrix}\left(\begin{matrix}1\\5\end{matrix}-\begin{matrix}6\\2\end{matrix}\right)+\begin{vmatrix}4\,5\\5\,6\end{vmatrix}\left(\begin{matrix}1\\4\end{matrix}-\begin{matrix}6\\3\end{matrix}\right)$$

$$-\begin{vmatrix}4\,6\\4\,5\end{vmatrix}\left(\begin{matrix}2\\6\end{matrix}-\begin{matrix}5\\1\end{matrix}\right)+\begin{vmatrix}4\,6\\4\,6\end{vmatrix}\left(\begin{matrix}2\\5\end{matrix}-\begin{matrix}5\\2\end{matrix}\right)-\begin{vmatrix}4\,6\\5\,6\end{vmatrix}\left(\begin{matrix}2\\4\end{matrix}-\begin{matrix}5\\3\end{matrix}\right)$$

$$+\begin{vmatrix}5\,6\\4\,5\end{vmatrix}\left(\begin{matrix}3\\6\end{matrix}-\begin{matrix}4\\1\end{matrix}\right)-\begin{vmatrix}5\,6\\4\,6\end{vmatrix}\left(\begin{matrix}3\\5\end{matrix}-\begin{matrix}4\\2\end{matrix}\right)+\begin{vmatrix}5\,6\\5\,6\end{vmatrix}\left(\begin{matrix}3\\4\end{matrix}-\begin{matrix}4\\3\end{matrix}\right),$$

each line of which may again be condensed by substituting for it a determinant of the third order, so that we shall have finally

$$\begin{vmatrix}4\,5\,1\\4\,5\,6\end{vmatrix}-\begin{vmatrix}4\,5\,6\\4\,5\,1\end{vmatrix}+\begin{vmatrix}4\,2\,6\\4\,5\,6\end{vmatrix}-\begin{vmatrix}4\,5\,6\\4\,2\,6\end{vmatrix}+\begin{vmatrix}3\,5\,6\\4\,5\,6\end{vmatrix}-\begin{vmatrix}4\,5\,6\\3\,5\,6\end{vmatrix}$$

$$=\begin{vmatrix}\genfrac{}{}{0pt}{}{4}{4} & \genfrac{}{}{0pt}{}{4}{5} & \genfrac{}{}{0pt}{}{4}{6}\\ \genfrac{}{}{0pt}{}{5}{4} & \genfrac{}{}{0pt}{}{5}{5} & \genfrac{}{}{0pt}{}{5}{6}\\ \genfrac{}{}{0pt}{}{1}{4}-\genfrac{}{}{0pt}{}{6}{3} & \genfrac{}{}{0pt}{}{1}{5}-\genfrac{}{}{0pt}{}{6}{2} & \genfrac{}{}{0pt}{}{1}{6}-\genfrac{}{}{0pt}{}{6}{1}\end{vmatrix}+\begin{vmatrix}\genfrac{}{}{0pt}{}{4}{4} & \genfrac{}{}{0pt}{}{4}{5} & \genfrac{}{}{0pt}{}{4}{6}\\ \genfrac{}{}{0pt}{}{2}{4}-\genfrac{}{}{0pt}{}{5}{3} & \genfrac{}{}{0pt}{}{2}{5}-\genfrac{}{}{0pt}{}{5}{2} & \genfrac{}{}{0pt}{}{2}{6}-\genfrac{}{}{0pt}{}{5}{1}\\ \genfrac{}{}{0pt}{}{6}{4} & \genfrac{}{}{0pt}{}{6}{5} & \genfrac{}{}{0pt}{}{6}{6}\end{vmatrix}+\begin{vmatrix}\genfrac{}{}{0pt}{}{3}{4}-\genfrac{}{}{0pt}{}{4}{3} & \genfrac{}{}{0pt}{}{3}{5}-\genfrac{}{}{0pt}{}{4}{2} & \genfrac{}{}{0pt}{}{3}{6}-\genfrac{}{}{0pt}{}{4}{1}\\ \genfrac{}{}{0pt}{}{5}{4} & \genfrac{}{}{0pt}{}{5}{5} & \genfrac{}{}{0pt}{}{5}{6}\\ \genfrac{}{}{0pt}{}{6}{4} & \genfrac{}{}{0pt}{}{6}{5} & \genfrac{}{}{0pt}{}{6}{6}\end{vmatrix}.$$

When $\genfrac{}{}{0pt}{}{1}{4}=\genfrac{}{}{0pt}{}{6}{3},\ \genfrac{}{}{0pt}{}{1}{5}=\genfrac{}{}{0pt}{}{6}{2},\ \ldots,$—that is to say, when the elements of $\begin{vmatrix}1\,2\,3\\4\,5\,6\end{vmatrix}$ are in order identical with those of $\begin{vmatrix}6\,5\,4\\3\,2\,1\end{vmatrix}$,—the right-hand side vanishes, and the theorem degenerates into the simpler one which suggested it.

(10) The corresponding theorem in connection with

$$\begin{vmatrix}1\,2\,3\,4\,5\,6\,7\,8\\1\,2\,3\,4\,5\,6\,7\,8\end{vmatrix}$$

is readily seen to be

$$\begin{vmatrix}5\,6\,7\,1\\5\,6\,7\,8\end{vmatrix}+\begin{vmatrix}5\,6\,2\,8\\5\,6\,7\,8\end{vmatrix}+\begin{vmatrix}5\,3\,7\,8\\5\,6\,7\,8\end{vmatrix}+\begin{vmatrix}4\,6\,7\,8\\5\,6\,7\,8\end{vmatrix}$$

$$-\begin{vmatrix}5\,6\,7\,8\\5\,6\,7\,1\end{vmatrix}-\begin{vmatrix}5\,6\,7\,8\\5\,6\,2\,8\end{vmatrix}-\begin{vmatrix}5\,6\,7\,8\\5\,3\,7\,8\end{vmatrix}-\begin{vmatrix}5\,6\,7\,8\\4\,6\,7\,8\end{vmatrix}$$

$$= \begin{vmatrix} {5 \atop 5} & {5 \atop 6} & {5 \atop 7} & {5 \atop 8} \\ {6 \atop 5} & {6 \atop 6} & {6 \atop 7} & {6 \atop 8} \\ {7 \atop 5} & {7 \atop 6} & {7 \atop 7} & {7 \atop 8} \\ {1 \atop 5} - {8 \atop 4} & {1 \atop 6} - {8 \atop 3} & {1 \atop 7} - {8 \atop 2} & {1 \atop 8} - {8 \atop 1} \end{vmatrix} + \begin{vmatrix} {5 \atop 5} & {5 \atop 6} & {5 \atop 7} & {5 \atop 8} \\ {6 \atop 5} & {6 \atop 6} & {6 \atop 7} & {6 \atop 8} \\ {2 \atop 5} - {7 \atop 4} & {2 \atop 6} - {7 \atop 3} & {2 \atop 7} - {7 \atop 2} & {2 \atop 8} - {7 \atop 1} \\ {8 \atop 5} & {8 \atop 6} & {8 \atop 7} & {8 \atop 8} \end{vmatrix}$$

$$+ \begin{vmatrix} {5 \atop 5} & {5 \atop 6} & {5 \atop 7} & {5 \atop 8} \\ {3 \atop 5} - {6 \atop 4} & {3 \atop 6} - {6 \atop 3} & {3 \atop 7} - {6 \atop 2} & {3 \atop 8} - {6 \atop 1} \\ {7 \atop 5} & {7 \atop 6} & {7 \atop 7} & {7 \atop 8} \\ {8 \atop 5} & {8 \atop 6} & {8 \atop 7} & {8 \atop 8} \end{vmatrix} + \begin{vmatrix} {4 \atop 5} - {5 \atop 4} & {4 \atop 6} - {5 \atop 3} & {4 \atop 7} - {5 \atop 2} & {4 \atop 8} - {5 \atop 1} \\ {6 \atop 5} & {6 \atop 6} & {6 \atop 7} & {6 \atop 8} \\ {7 \atop 5} & {7 \atop 6} & {7 \atop 7} & {7 \atop 8} \\ {8 \atop 5} & {8 \atop 6} & {8 \atop 7} & {8 \atop 8} \end{vmatrix};$$

and the general theorem is

If the symbol

$$\sum_{\mu=1,2,\ldots,n} \begin{vmatrix} n+1, n+2, \ldots, n-\mu+1, \ldots, 2n \\ n+1, n+2, \ldots, n+\mu \quad , \ldots, 2n \end{vmatrix}$$

stand for the sum of the n *determinants whose column-indices are in every case* $n+1, n+2, \ldots, 2n$ *and whose row-indices are the same except that for one of them there has been substituted its defect from* $2n+1$; *and if*

$$\begin{vmatrix} n+1, n+2, \ldots, n-\overset{*}{\mu}+1, \ldots, 2n \\ n+1, n+2, \ldots, n+\mu \quad , \ldots, 2n \end{vmatrix}$$

be taken to indicate that in the determinant

$$\begin{vmatrix} n+1, n+2, \ldots, n-\mu+1, \ldots, 2n \\ n+1, n+2, \ldots, n+\mu \quad , \ldots, 2n \end{vmatrix}$$

each element ${\alpha \atop \beta}$ *of the* μ^{th} *row is to be diminished by the element* ${2n+1-\alpha \atop 2n+1-\beta}$; *then,*

$$\sum_{\mu=1,\ldots,n}\begin{vmatrix} n+1, n+2, \ldots, n-\mu+1, \ldots, 2n \\ n+1, n+2, \ldots, n+\mu \quad , \ldots, 2n \end{vmatrix}$$

$$-\sum_{\mu=1,\ldots,n}\begin{vmatrix} n+1, n+2, \ldots, n+\mu \quad , \ldots, 2n \\ n+1, n+2, \ldots, n-\mu+1, \ldots, 2n \end{vmatrix}$$

$$= \sum_{\mu=1,\ldots,n}\begin{vmatrix} n+1, n+2, \ldots, n-\overset{*}{\mu}+1, \ldots, 2n \\ n+1, n+2, \ldots, n+\mu \quad , \ldots, 2n \end{vmatrix}.$$

(11) From this there follows, exactly as before, a variant form of the enunciation of the less general theorem with which we started, viz.,

In connection with an even-ordered determinant $\begin{vmatrix} 1\,2\,3 \ldots 2n \\ 1\,2\,3 \ldots 2n \end{vmatrix}$ *which is such that the elements of* $\begin{vmatrix} 1 & 2 & & n \\ n+1, & n+2, & \ldots, & 2n \end{vmatrix}$ *are in order identical with those of* $\begin{vmatrix} 2n, & \ldots, & n+2, & n+1 \\ n, & \ldots, & 2, & 1 \end{vmatrix}$ *we have*

$$\sum_{\mu=1,\ldots,n}\begin{vmatrix} n+1, n+2, \ldots, n-\mu+1, \ldots, 2n \\ n+1, n+2, \ldots, n+\mu \quad , \ldots, 2n \end{vmatrix}$$

$$-\sum_{\mu=1,\ldots,n}\begin{vmatrix} n+1, n+2, \ldots, n+\mu \quad , \ldots, 2n \\ n+1, n+2, \ldots, n-\mu+1, \ldots, 2n \end{vmatrix} = 0.$$

The advantage of this form of enunciation, again, is that it indicates the limited amount of centro-symmetry which is necessary for the validity of *one* of my 1888 identities, and shows that the number of such identities possible, when the centro-symmetry is complete and the parent determinant is of the $(2n)^{\text{th}}$ order, is $C_{2n,\,n}$.

Note on the Activity of the Saliva in Diseased Conditions of the Body. By W. G. Aitchison Robertson, M.D., D.Sc., F.R.C.P.E.

(Read February 19, 1900.)

The investigation was undertaken to find out in what way the activity of the salivary ferment varied in different diseased conditions of the body.

In order to eliminate the fallacy which might arise from the hourly variation in the diastatic power of the secretion, the experiments were always performed at the same hour each evening. Each individual was made to wash his mouth out thoroughly with slightly warm water, and, during the succeeding half hour, all the saliva which he secreted was received into a vessel and measured. Two cubic centimetres of the saliva were then mixed with ten cubic centimetres of starch mucilage at the temperature of 38° C., and the mixture was then kept at this temperature for ten minutes. At the end of this period the condition of the starch present was noted, and further action of the ferment was prevented by rapidly boiling the mixture. The amount of sugar which had been formed by the ptyalin was then estimated by titration against standard Fehling's solution.

Above one hundred cases of disease of various kinds were investigated, in order to see if the activity of the salivary ferment had undergone any change.

Gastro-intestinal Disorders.—Twenty-one cases were examined. The average amount of sugar formed in these was 0·089 gramme (the normal average being taken as 0·080). In chronic gastric catarrh this figure varied from 0·078 to 0·1 gramme. In acid dyspepsia the amount of sugar formed is above the healthy average, while in ulceration of the stomach, the amount is generally only slightly below the normal average. In dilatation of the stomach, the salivary ferment was found to be almost absent, or at least inactive. In cirrhosis of the liver the amount of sugar is not reduced, and in some cases it is greatly increased.

Pulmonary Diseases.—In these diseases generally the salivary

ferment is fairly active, and on an average 0·087 gramme of sugar is formed. In phthisis the ferment is present in normal amount, and in pneumonia the amylolytic power of the saliva is above the normal during the period preceding the crisis, but lower after this event.

Heart Diseases.—In the large group of heart cases the saliva retains its usual composition, and the amount of sugar formed hovers at or about the normal limits.

Nervous Diseases.—A larger proportion of subnormal cases occurred in this group, fully 41 per cent., giving a proportion of sugar lower than the normal average. In one case of cerebral tumour, the saliva, though copious in amount, contained practically no converting ferment; whereas in a case of locomotor ataxia, though the secretion was equally copious, the salivary enzyme produced the large amount of 0·111 gramme.

Hæmopoietic System.—Of three cases of Addison's disease examined, the saliva of two showed marked deficiency in diastatic power, while the third exceeded the normal limit.

Renal Diseases.—The group of diseases of the kidneys shows generally a lower average than the normal. In 55·5 per cent. the quantity of sugar produced was considerably below the average.

In diabetes the saliva has a very active converting power. In three out of four cases examined, the average amount of sugar formed was much above the standard figure.

In simple anæmia the converting ferment seems to be present in the saliva in its normal amount. If, however, the anæmia be associated with dyspepsia, the average is subnormal.

In *Sub-acute* and *Chronic Rheumatism* the ferment exists in its normal amount.

In general febrile conditions the secretion of saliva is greatly reduced in amount, and this reduction increases *pari passu* with the increase in temperature. This scanty secretion seems, however, to possess increased amylolytic power.

Quantity of Saliva Secreted.

Gastric Diseases.—In most cases of acid dyspepsia the amount of saliva secreted is above the normal. In chronic gastric catarrh

the quantity is hardly up to the average, and the same is seen in ulceration of the stomach. In those cases associated with diarrhœa or ascites the secretion is often far below the normal.

Pulmonary Diseases.—In bronchitis and in the early stages of pneumonia the secretion of saliva is generally up to the full average, and may even exceed it. In chronic phthisis the secretion of saliva is always very scanty.

Cardiac Diseases.—The salivary secretion is almost constantly diminished when the heart affection is of a grave character.

Nervous Diseases.—In affections of the cord, the quantity of saliva secreted reaches, and even surpasses, the average amount. In the case of cerebral tumours the reverse is found, however.

Renal Diseases.—In chronic Bright's disease, the secretion is generally scanty.

In simple anæmia, in the chronic forms of rheumatism and in Addison's disease, the amount of saliva secreted is subnormal.

In fevers generally, when the temperature is at all high, the secretion is lessened in amount, though the amylolytic power is increased.

In many cases where the secretion is scanty the diastatic power is likewise feeble, and, on the contrary, where the secretion is copious its proteolytic power is also great.

On *Tetrabothrium torulosum* and *Tetrabothrium auriculatum*. By **Dr O. von Linstow**, Göttingen. *Communicated by* Sir John Murray, K.C.B.

(Read May 21, 1900.)

In my report on the Entozoa, brought home by the "Challenger" expedition, I described two new species as *Tetrabothrium torulosum* (from *Diomedea brachyura*) and *Tetrabothrium auriculatum* (from *Thalassœca glacialis* and *Daption capensis*).* My descriptions have recently been subjected to adverse criticism by Fuhrmann,† who maintains that these two species do not belong to the genus *Tetrabothrium*, but to the genus *Prosthecocotyle*; that *Tetrabothrium auriculatum* is identical with *Tetrabothrium* (*Amphoterocotyle*) *elegans-heteroclitum*, Diesing; that these two Cestodes are not *Tetrabothria* but typical *Tæniæ*; that my drawing of the scolex of *Tetrabothrium torulosum* does not correspond with the actual relations; and that my representation of the masculine genital organs of *Tetrabothrium auriculatum* is inaccurate.

With reference to the genus to which the two species are to be referred, I certainly could not place them in the genus *Prosthecocotyle*, for my description appeared in the year 1888, while *Prosthecocotyle* of Monticelli, and the synonymous genus *Bothridiotænia* of Lönnberg, were founded in 1896. The genus *Tetrabothrium* was known to me by two species: *Tetrabothrium cylindraceum*, Rud., and *Tetrabothrium macrocephalum*, Rud., the two typical species upon which Rudolphi, about a hundred years ago, founded the genus *Tetrabothrium*, and as the two species in question agreed in all essential points with the two species described by Rudolphi, I placed them in the genus *Tetrabothrium*.

Tetrabothrium is not related to the Bothriocephalids but to the

* "*Challenger*" *Reports*, Zoology, vol. xxiii. part lxxi. pp. 14, 15, Pl. II. figs. 16–20, 1888.

† *Zool. Anzeiger*, No. 561, pp. 385–388, 1898; *Proc. Roy. Soc. Edin.*, vol. xxii. pp. 641–651, 1899.

Tæniæ; the scolex carries four large sucking cups, which touch each other with their edges completely or partly, and are drawn in front or behind into an angle; the proglottides are short, the genital openings are marginal and unilateral, and pass into a genital sinus with strong muscular walls, on the inner side of which lies the round cirrus-pouch. The vas deferens is rolled up in numerous coils; the parenchyma muscles, especially the longitudinal muscles, are strongly developed, and on each side two longitudinal vessels join together; the ovarium is strongly developed, the small oviduct lying before it. This is briefly the diagnosis of the genus, as given by me * when describing *Tetrabothrium cylindraceum*, and it corresponds perfectly with Monticelli's genus *Prosthecocotyle* and with Lönnberg's *Bothridiotænia*; the genera *Prosthecocotyle*, Monticelli, and *Bothridiotænia*, Lönnberg, are thus synonyms of *Tetrabothrium*, Rudolphi, and as the last-mentioned name has priority, Fuhrmann is mistaken in believing that the two species described by me belong to the genus *Prosthecocotyle*, for they must be placed in the genus *Tetrabothrium*.

Fuhrmann, in his description of *Tetrabothrium*, says:—"The interpretations of the male sexual apparatus given by Linstow are inexact," yet my description of the male organs is limited to the sentence:—"The cylindrical cirrus is protruded to a length of 0·082 mm., and is 0·016 mm. in breadth," which is perfectly correct. Fuhrmann further says that my representation of the scolex of *Tetrabothrium torulosum* does not correspond with the actual relations, but I maintain, on the contrary, that having prepared my drawings carefully with the aid of the drawing apparatus, it must be assumed that they really show the actual relations. I forbear to express an opinion regarding Fuhrmann's drawings.

Fuhrmann finally says that my drawing of *Tetrabothrium auriculatum* is identical with Diesing's *Tetrabothrium heteroclitum*,† but neither from the description nor the drawing can this identity be recognised. Diesing says:—"Caput clavatum, bothriis lateralibus oblongis prominulis, limbo tumidulis, antrorsum convergentibus,"

* *Centralblatt f. Bakter., Parasitenk., und Infektionskrankh.*, Abth. I. Bd. xxvii. pp. 365–6, Jena, 1900.

† *Denksch. math.-nat. Cl. d. k. Akad. d. Wissensch.*, Wien, Bd. xii. p. 28, tab. II. figs. 25–37, 1856.

and in figs. 27–29 he represents the scolex with converging rounded edges towards the front; the protruding angles at the front edge, so characteristic in *Tetrabothrium auriculatum*, are quite absent, so that a relationship cannot be assumed.

I must therefore reject all Fuhrmann's criticisms as completely unfounded and superfluous.

Contributions to the Craniology of the People of India. Part II.—The Aborigines of Chútá Nágpúr, of the Central Provinces and the people of Orissa. By Professor Sir William Turner, F.R.S.

(Read July 2, 1900.)

(*Abstract.*)

This part of my memoir on the crania of the people of India is especially occupied with a description of the hill tribes in the Lower provinces of Bengal and the Central provinces. It is based on an examination of a number of crania, the majority of which were placed at my disposal by the authorities of the Indian Museum, Calcutta. Some belonged to tribes speaking dialects of the Kolarian group of languages; others of the Dravidian group.

The Dravidians were represented by skulls of the Gond, Oráon, Pahária, Kharwár, Khand, Nágesar, Korwá and Bhuiya tribes; the Kolarians by skulls of the Múnda or Ho, Bhumij and Turi tribes.

In addition, a few skulls of the Ahír-Goálá, Kámár, Lohár and Teli castes, and two crania ascribed to the tribe of Juangs came under observation. A number of skulls from Orissa, belonging to Uriyá-speaking people, were also described.

The skulls of the Dravidians and Kolarians were compared with each other, with the object of testing their bearing on the opinion expressed by Mr H. H. Risley, based upon observations on, and measurements of, about 6000 living persons, that the differences between these two groups are only linguistic, and do not represent differences in physical type. The comparison was based on the study of seventeen Dravidian skulls and nineteen belonging to Kolarian tribes, and the conclusion was drawn that they corresponded in essential particulars. In both, the form and proportions of the cranium were dolichocephalic; the anterior nares were platyrhine, or in the higher term of the mesorhine group; the

upper jaw was orthognathous, only one specimen was prognathous; as a rule the orbit was low or microseme; the palato-alveolar arch was brachyuranic, and the face was short in relation to its width. The cranial characters therefore supported the conclusions drawn by Mr Risley from the examination of living persons.

The skulls of the Kámár, Ahír-Goálá and Teli castes also possessed Dravidian characters. The Lohár skull again, from its leptorhine nasal index, showed an Aryan feature.

The crania of the Uriyá-speaking people had mixed characters, as if there had been an intermingling of Aryans with Hinduised aborigines, and possibly traces of a brachycephalic stock.

A comparison was made between the Dravidian skulls and those of the aboriginal Australians. Although both are dolichocephalic and platyrhine, yet in many other respects, more especially in their greater absolute length, their more roof-shaped crania, the degree of projection of the glabella, the depressed nasion, the prognathic upper jaw, the elongated palate, and the coarse, large teeth, the Australians differed from the Dravidians in important characters.

The Action of Silver Salts on Solution of Ammonium Persulphate. By Hugh Marshall, D.Sc. (With a Plate.)

(Read February 5, 1900.)

Although the action of potassium persulphate on silver nitrate solution was one of the first persulphate reactions observed (vol. xviii. p. 64), I had not until lately paid any special attention to the behaviour of the ammonium salt in this respect. It appears, however, that in the latter case there are additional actions of great interest, not possible with the potassium salt. A general description of these will be given now, but there are still some points deserving of further investigation.

When solutions of potassium persulphate and silver nitrate are mixed, a black precipitate slowly forms, and this precipitate exhibits all the characteristics of silver peroxide. Apparently silver persulphate (which we may assume to be formed, to a certain extent, by double decomposition) is decomposed by water, like so many other silver salts of sulphur acids, by abstraction of SO_3 to form sulphuric acid.

$$Ag_2S_2O_8 + 2H_2O = 2H_2SO_4 + Ag_2O_2.$$

In course of time the precipitate decomposes and dissolves with evolution of oxygen.

When ammonium persulphate solution is mixed with silver nitrate solution a similar result is seen, but only to a slight extent. Although there is very little deposition of peroxide, there is, however, a considerable amount of decomposition, as shown by the formation of sulphate and free acid in the liquid. If ammonia is added to the mixed salt solutions there is no separation of peroxide, but there is a much more rapid formation of sulphate accompanied with brisk effervescence. These reactions can be easily followed by starting with a pure persulphate solution and adding barium nitrate along with the other reagents.

As it is known that silver peroxide oxidises ammonia to nitrogen,

the above-mentioned effervescence was presumably due to the escape of the latter gas. To test this, ammonium persulphate in considerable quantity was dissolved in strong ammonia solution; a small flask was filled almost completely with the solution, some silver nitrate added, and an india-rubber stopper with delivery-tube fitted to the flask, so that the evolved gas might be collected in a vessel over water. The evolution of gas began at once and increased rapidly; at the same time the temperature of the liquid rose, and soon the action became violent. Ultimately the stopper and fittings were driven out, and most of the liquid blown out of the flask.

The first quantities of gas had been allowed to escape, after which sufficient for examination was secured before the unexpectedly sudden termination of the experiment. The sample contained a mere trace of oxygen, the presence of which was almost certainly due to the method of collection.

The quantity of silver salt employed in this experiment would amount to only a few centigrams, and it is therefore evident that the silver must oscillate very rapidly between the two stages of oxidation in order to cause such rapid decomposition. Apparently we have here an admirable example of a 'catalytic action,' in which the part played by the catalytic agent may be considered as definitely known. The final result is expressible by the simple equation—

$$3(NH_4)_2S_2O_8 + 8NH_3 = 6(NH_4)_2SO_4 + N_2$$

leaving the silver compound entirely out of account, but there seems no reason to doubt that the action takes place in the manner and stages indicated.

The experiment is one very well suited for class demonstration, and is exceedingly simple. Dissolve a considerable quantity of ammonium persulphate in concentrated ammonia solution, and place the solution in a tall beaker or jar. Add a small quantity of silver nitrate solution; the evolution of nitrogen begins at once, and soon the temperature rises so high that large quantities of ammonia gas also escape, causing the liquid to boil over; the result is not nearly so striking if dilute ammonia solution is employed.

The decomposition of an ordinary aqueous solution of ammonium

persulphate in presence of silver salts is much slower than the above, and it appeared interesting to get some idea of the rate at which it took place as compared with that of a solution free from silver. So far it had been assumed that the products would simply be oxygen and ammonium hydrogen sulphate.

Twelve grams of recrystallised, but not quite pure, ammonium persulphate were dissolved in water at 20°C., and, after the addition of 0·0425 gm. of silver nitrate (corresponding ultimately to a milligram equivalent per litre of solution), the solution was diluted to 250 c.c. The solution was kept in a thermostat at the temperature stated. From time to time, 5 c.c. were withdrawn and titrated with fifth-normal alkali solution, using methyl orange as indicator. In the earlier titrations, when there still remained a good deal of undecomposed persulphate, the indicator became rapidly bleached, and in each of these cases it was found advisable to repeat the determination, adding the indicator only when the neutral point was nearly reached, as known from the first determination.

Time.	Vol. of 2N alkali for 5 c.c. of solution.	Time.	Vol. of ·2N alkali for 5 c.c. of solution.
0d. 5h.	0·7 c.c.	8d. 5½h.	11·25 c.c.
1 1½	3·15	9 6½	11·6
2 2	5·45	10 6	11·8
3 3	7·2	13 6½	12·3
4 4	8·5	16 6	12·55
5 5	9·5	35 2	12·75
6 7	10·3		

These results are plotted in the figure (see Plate), the curve showing the increase of acidity as the experiment progressed.

It soon became evident that the reaction was not taking place in the way imagined. Allowing for the small quantity of sulphate in the sample of salt employed, the persulphate solution was slightly over 0·4 normal. Therefore it should ultimately have produced a slightly more than 0·4 normal acid solution, assuming the final decomposition to be expressible by the equation—

$$2(NH_4)_2S_2O_8 + 2H_2O = 2(NH_4)_2SO_4 + 3H_2SO_4 + O_2.$$

The quantity of alkali solution required for 5 c.c. of the liquid should therefore have approached a limit of slightly over 10 c.c.

Instead of that, the limit was clearly considerably higher, about 12 or 13. (The number actually obtained was 12·75.) Further, at no stage was there any evolution of gas observable, even on shaking the liquid, notwithstanding the large quantity of salt decomposed.

The only reasonable assumption to be made was that the oxygen, instead of being liberated, was being used up to oxidise the hydrogen of ammonium, probably also the nitrogen—otherwise there should still have been a considerable evolution of gas, as shown by the equation—

$$3(NH_4)_2S_2O_8 = 2(NH_4)_2SO_4 + 4H_2SO_4 + N_2.$$

This would give an increased acidity of one third, making the limit about 13·5. On the other hand, assuming nitric acid to be the oxidation product, as shown by the equation

$$8(NH_4)_2S_2O_8 + 6H_2O = 7(NH_4)_2SO_4 + 9H_2SO_4 + 2HNO_3,$$

the increased acidity would be only one fourth, giving a limit of slightly over 12·5.* As a matter of fact, the liquid was found to give a very well marked nitric acid reaction, although the proportion of silver nitrate originally added was far too small to be appreciable in a small quantity of the solution by means of the usual test for nitric acid. The matter was put beyond all doubt by heating about a gram of ammonium persulphate with solution of silver sulphate. There was only a slight evolution of gas, although the liquid was heated nearly to boiling, and the resulting liquid contained so much nitric acid that there was a quite considerable evolution of nitric oxide on treatment with ferrous sulphate and sulphuric acid.

The quantitative experiment was commenced merely to obtain a rough idea of the increased rate of decomposition, and was not carried out in a very strict manner, the titrations being performed

* Writing the three equations in comparable terms we have:—

$$(1)\ 24(NH_4)_2S_2O_8 + 24H_2O = 24(NH_4)_2SO_4 + 24H_2SO_4 + 12O_2.$$
$$(2)\ 24(NH_4)_2S_2O_8 = 16(NH_4)_2SO_4 + 32H_2SO_4 + 8N_2.$$
$$(3)\ 24(NH_4)_2S_2O_8 + 18H_2O = 21(NH_4)_2SO_4 + 27H_2SO_4 + 6HNO_3.$$

These give respectively $48H^{\cdot}$, $64H^{\cdot}$, and $60H^{\cdot}$, for the same quantity of persulphate, corresponding to the ratio:—1, 1·33, 1·25.

at varying intervals of time. A more systematic series of experiments, under various conditions, may be expected to yield interesting results. The results expressed by the above curve are nevertheless such as to show that, for moderate concentrations, the quantity of salt decomposed in a given time is practically proportional to the quantity of persulphate present. As the reaction is not a unimolecular one, this would seem to indicate that one of the intermediate stages takes place much more slowly than the others.

The spontaneous decomposition of an aqueous solution of ammonium persulphate takes place at a far slower rate than the above-noted one. After the lapse of four weeks, under the same conditions of concentration and temperature, a pure solution of the salt had decomposed to such an extent that 5 c.c. required only 0·5 c.c. of alkali solution. In this case the bleaching of the methyl orange indicator was also much less rapid, and caused no practical inconvenience.

By employing solutions of greater concentration (as regards both persulphate and silver) and a higher temperature, considerable quantities of nitric acid may be produced in this way. If the temperature is kept very high there is a fair amount of other decomposition, oxygen mixed with ozone being evolved in considerable quantity if the liquid is boiled.

There are probably many other reactions which may be either brought about or accelerated by the catalytic action of silver compounds in presence of persulphate. We have such a case in the oxidation of methyl orange, already noted. A similar one is presented by the oxidation of indigo. If a solution of ammonium persulphate is coloured by means of indigo, then divided into two portions, and a drop of silver nitrate solution added to one of them, that one will be found to be decolorised much faster than the other.

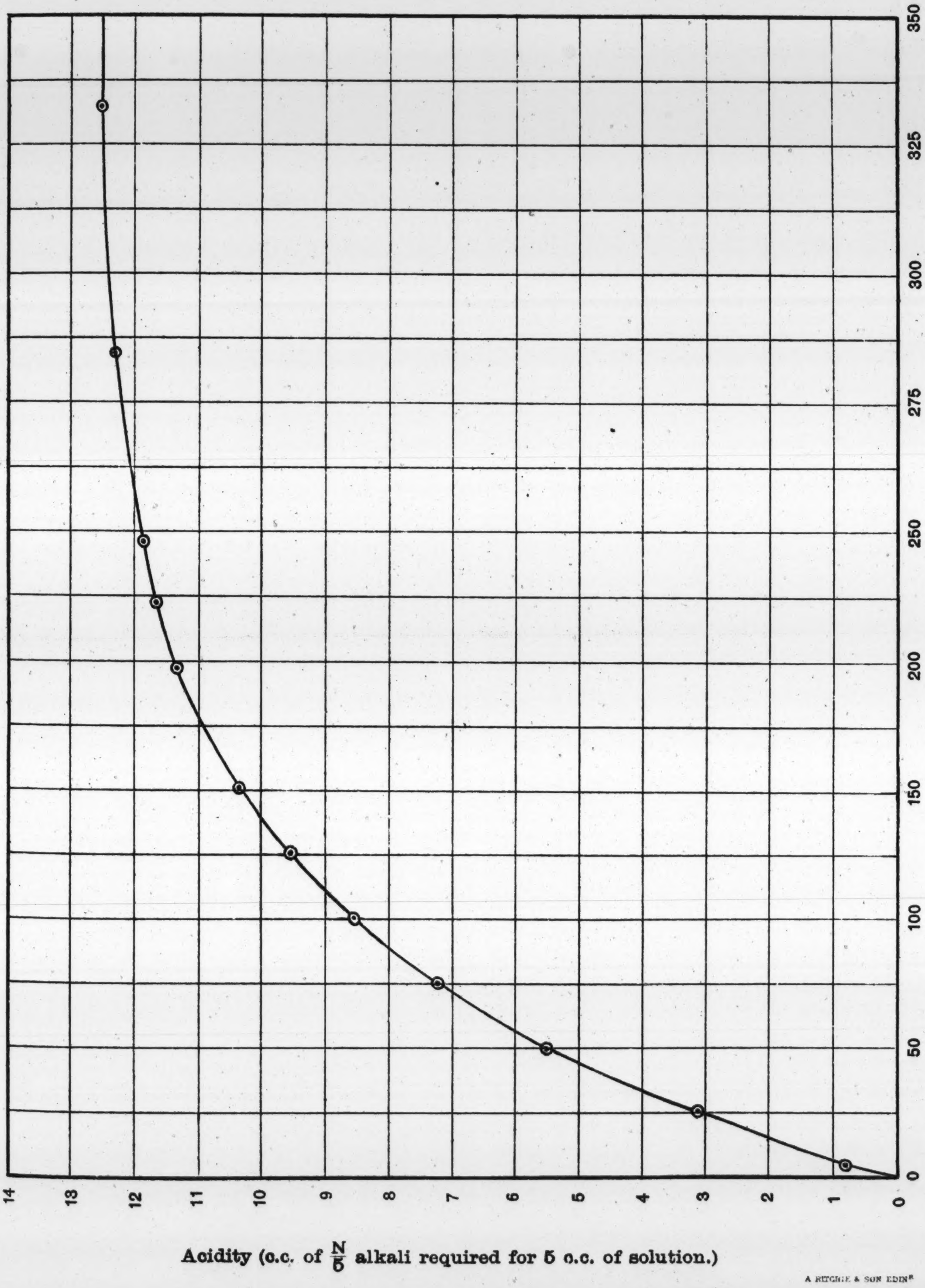
A still more remarkable example is provided by the oxidation of a chromic salt to chromic acid *in acid solution.* If solution of, say, chrome alum is heated with pure ammonium persulphate no change is observable beyond the usual one from purple colour to green. If, however, a drop of silver nitrate solution is also added, and the mixture gently warmed, the colour changes to green and then

to bright yellow, and ultimately the solution is found to contain chromic acid and no chromic salt.

Ammonium persulphate is now made and employed technically on a considerable scale. Possibly the employment of small quantities of silver compounds in conjunction with it may extend its applicability as an oxidising agent to cases where by itself it would be ineffective.

There is another point of interest in connection with the use of ammonium persulphate solution as a 'reducer' in photography. A solution which has been once used for this purpose is bound to contain sufficient silver salt to accelerate enormously the rate of decomposition and render the solution very soon unfit for use, although in its unused condition it might be kept for a considerable time without undergoing decomposition to a serious extent. It is also possible that the metallic silver of the film is more rapidly attacked once there is some of the peroxidic compound present in the solution. It has been stated, indeed, that pure solution of ammonium persulphate does not attack the film, and that the action only commences once a small quantity of ozone has been formed by decomposition. If that is so, then probably the addition of a small quantity of silver nitrate solution to a 'reducer' freshly prepared from pure ammonium persulphate would make it immediately active.

MARSHALL: ACTION OF SILVER SALTS ON SOLUTION OF AMMONIUM PERSULPHATE.



A. RITCHIE & SON EDIN^R

Hyperbolic Quaternions. By Alexander Macfarlane, Lehigh University, South Bethlehem, Pennsylvania. (With a Plate.)

(Read July 16, 1900.)

It is well known that quaternions are intimately connected with spherical trigonometry, and in fact they reduce that subject to a branch of algebra. The question is suggested whether there is not a system of quaternions complementary to that of Hamilton, which is capable of expressing trigonometry on the surface of the equilateral hyperboloids. The rules of vector-analysts are approximately complementary to those of quaternions. In this paper I propose to show how they can be made completely complementary, and that, when so rectified, they yield the hyperbolic counterpart of the spherical quaternions.

The celebrated rules discovered by Hamilton are :—

$$\begin{array}{lll} i^2 = -1 & j^2 = -1 & k^2 = -1 \\ ij = k & jk = i & ki = j \\ ji = -k & kj = -i & ik = -j. \end{array}$$

This is the statement of the rules as enunciated by Hamilton; it supposes an order of the symbols from right to left. When the order is changed to that from left to right, they become :—

$$\begin{array}{lll} i^2 = -1 & j^2 = -1 & k^2 = -1 \\ ij = -k & jk = -i & ki = -j \\ ji = k & kj = i & ik = k. \end{array}$$

The rules used by vector-analysts are :—

$$\begin{array}{lll} i^2 = +1 & j^2 = +1 & k^2 = +1 \\ ij = k & jk = i & ki = j \\ ji = -k & kj = -i & ik = -j, \end{array}$$

and they suppose an order from left to right. They lead to products in which the manner of associating the factors is essential, in this respect differing from the rules of quaternions. Can they be modified so that the order of the factors will be preserved,

while the products become associative? I find that the desired modification is accomplished by introducing $\sqrt{-1}$ before the second and third sets. The rules then become

$$\begin{array}{lll} i^2 = +1 & j^2 = +1 & k^2 = +1 \\ ij = \sqrt{-1}k & jk = \sqrt{-1}i & ki = \sqrt{-1}j \\ ji = -\sqrt{-1}k & kj = -\sqrt{-1}i & ik = -\sqrt{-1}j. \end{array}$$

As the quaternion $i\,j\,k$ are quadrantal unit-vectors, they can be analysed into $\sqrt{-1}i_0$, $\sqrt{-1}j_0$, $\sqrt{-1}k_0$, where $i_0\,j_0\,k_0$ are unit-vectors.

The quaternion rules, modified for order, then become

$$(\sqrt{-1}i_0)(\sqrt{-1}i_0) = -1 \qquad (\sqrt{-1}j_0)(\sqrt{-1}j_0) = -1$$

$$(\sqrt{-1}k_0)(\sqrt{-1}k_0 = -1$$

$$(\sqrt{-1}i_0)(\sqrt{-1}j_0 = -\sqrt{-1}k_0 \qquad (\sqrt{-1}j_0)(\sqrt{-1}k_0) = -\sqrt{-1}i_0$$

$$(\sqrt{-1}k_0)(\sqrt{-1}i_0) = \sqrt{-1}j_0$$

$$(\sqrt{-1}j_0)(\sqrt{-1}i_0) = \sqrt{-1}k_0 \qquad (\sqrt{-1}k_0)(\sqrt{-1}j_0) = \sqrt{-1}i_0$$

$$(\sqrt{-1}i_0)(\sqrt{-1}k_0) = \sqrt{-1}j_0.$$

These rules are in perfect harmony with the vector rules when made associative as above; for, on dividing the left hand by $\sqrt{-1}$ $\sqrt{-1}$, and the right hand side by the equivalent $-$, they yield

$$\begin{array}{lll} i_0{}^2 = 1 & j_0{}^2 = 1 & k_0{}^2 = 1 \\ i_0 j_0 = \sqrt{-1}k_0 & j_0 k_0 = \sqrt{-1}i_0 & k_0 i_0 = \sqrt{-1}j_0 \\ j_0 i_0 = -\sqrt{-1}k_0 & k_0 j_0 = -\sqrt{-1}i_0 & i_0 k_0 = -\sqrt{-1}j_0. \end{array}$$

Let ρ denote any real unit axis; then $\rho^2 = 1$. Similarly for any imaginary unit axis $(\sqrt{-1}\rho)^2 = -1$. It is evident that $\rho^2 = 1$ is in nature a principle of reduction. But there is also the principle of reduction $\rho/\rho = 1$ or $\sqrt{-1}\rho/\sqrt{-1}\rho = 1$. This latter is a more absolute principle, and the reduction specified can be made at any time; whereas the former is legitimate only under certain conditions. The rules of the form $ij = \sqrt{-1}k$ are also principles of reduction of a relative nature.

A more general statement of these rules is as follows:—For any two real unit axes β and γ.

$$\beta\gamma = \cos\beta\gamma + \sin\beta\gamma\,\sqrt{-1}\,\overline{\beta\gamma}$$

where $\overline{\beta\gamma}$ denotes in the simplest case the axis perpendicular to β and γ, but more correctly the axis conjugate to the plane of

β and γ. Similarly for any two imaginary axes $\sqrt{-1}\beta$ and $\sqrt{-1}\gamma$ $(\sqrt{-1}\beta)(\sqrt{-1}\gamma) = -\cos\beta\gamma - \sin\beta\gamma\ \sqrt{-1}\overline{\beta\gamma}$.

I proceed now to apply these principles to the investigation of the fundamental theorems of hyperboloidal trigonometry. I shall consider only the hyperboloid of equal axes, but the results can easily be extended to the general hyperboloid.

On account of the symmetry of the sphere with respect to its centre, spherical quaternions are independent of rectangular axes. It is otherwise with hyperboloidal quaternions, for the equilateral hyperboloid has an axis of revolution. In order to treat of trigonometry on the hyperboloid, it is necessary first to treat the trigonometry of the sphere with reference to the same axis of revolution. In the figure (fig. 1) OA is the axis of revolution, and the surfaces considered are those generated by the circle and by the equilateral hyperbolas. From this point of view the circle appears as consisting of a real part PQ corresponding to the real hyperbola P′Q′, and an imaginary part QR corresponding to the imaginary hyperbola Q′R′. Consequently the sphere appears broken up into a double sheet traced out by PQ and RS, and a single sheet traced out by QR.

The algebraic expression for a circular angle is $e^{b\sqrt{-1}}$. As the axis of the plane is not specified, the denotation of the expression is necessarily limited to angles in a constant plane. Let β be introduced to denote the axis, then $e^{b\sqrt{-1}\beta}$ is the proper expression for an angle in any plane. We have

$$e^{b\sqrt{-1}\beta} = 1 + b\sqrt{-1}\beta + \frac{(b\sqrt{-1}\beta)^2}{2\,!} + \frac{(b\sqrt{-1}\beta)^3}{3\,!} + .$$

Let the principle of reduction be introduced, which reduces $(\sqrt{-1}\beta)^2 = -1$; then the right hand member becomes

$$\begin{aligned}
&1 + b\sqrt{-1}\beta - \frac{b^2}{2\,!} - \frac{b^3}{3\,!}\sqrt{-1}\beta + \text{etc.}\\
&= 1 - \frac{b^2}{2\,!} + \frac{b^4}{3\,!} - \\
&\quad + \left(b - \frac{b^3}{3\,!} + \frac{b^5}{5\,!} - \right)\sqrt{-1}\beta\\
&\quad = \mathrm{SU}q + \mathrm{VU}q\\
&\quad = \cos b + \sin b\,(\sqrt{-1}\beta).
\end{aligned}$$

Note that the expression $SUq + VUq$ is not the complete equivalent of Uq; the binomial is a reduced equivalent. For, if β is variable, the result of differentiating $e^{b\sqrt{-1}\beta}$ will be different from the result of differentiating $\cos b + \sin b\ (\sqrt{-1}\beta)$.

If we enquire for the analogous expression for a hyperbolic angle, we find that there is none furnished by Algebra. It is not e^b, for

$$e^b = 1 + b + \frac{b^2}{2\,!} + \frac{b^3}{3\,!} +$$

and there is here no ground for breaking up the series into two components; all the terms are real, and so add directly. For the same reason it cannot be e^{-b}. But we know that

$$\cosh b = 1 + \frac{b^2}{2\,!} + \frac{b^4}{4\,!} +,$$

$$\sinh b = b + \frac{b^3}{3\,!} + \frac{b^5}{5\,!} +;$$

there must therefore be some proper way of expressing a hyperbolic angle by means of an exponential function. Try the effect of dropping $\sqrt{-1}$ from the circular expression $e^{b\sqrt{-1}\beta}$. We get

$$e^{b\beta} = 1 + b\beta + \frac{(b\beta)^2}{2\,!} + \frac{(b\beta)^3}{3\,!} + .$$

Now introduce the corresponding principle of reduction, namely, $\beta^2 = +1$; then

$$\begin{aligned} e^{b\beta} &= 1 + b\beta + \frac{b^2}{2\,!} + \frac{b^3}{3\,!}\beta + \\ &= 1 + \frac{b^2}{2\,!} + \frac{b^3}{3\,!} + \\ &\quad + (b + \frac{b^3}{3\,!} + \frac{b^5}{5\,!} +)\beta \\ &= SUq' + VUq' \end{aligned}$$

if q' denotes a hyperbolic quaternion. Hence it appears that $e^{b\beta}$ is the proper expression for the angle of an equilateral hyperbola.

It follows that the expression for the spherical quaternion is $re^{b\sqrt{-1}\beta}$, which, after expansion and reduction, gives the spherical complex quantity of the form $x + y\sqrt{-1}\beta$. Similarly the expression for the equilateral hyperbolic quaternion is $re^{b\beta}$, which, after expansion and reduction, gives the hyperbolic complex quantity of the form $x + y\beta$. In the former case we have $r = \sqrt{x^2 + y^2}$; in the latter, $r = \sqrt{x^2 - y^2}$. Suppose the objection

made, x may be equal to y, what then becomes of the modulus? The answer is, the cosine is then $\frac{x}{o}$, which shows that the angle is infinitely great, and this is the geometrical truth. Suppose that the objection is made, x may be less than y, what then becomes of the modulus? The modulus then takes on a form appropriate to the conjugate hyperbola, and by the hypothesis the angle lies in the conjugate hyperbola.

The above expression for a spherical quaternion has a resemblance to the *Drehstreckung* of Professor Klein. But r does not mean an expansion and $e^{b\sqrt{-1}\beta}$ a rotation; the former is a multiplier simply, and the latter a circular angle. The existence of the analogous expression $re^{b\beta}$, and the application of these expressions to develop the trigonometry of surfaces of the second order show that his theory of quaternions is inadequate, and the sphere of applicability which he assigns them too narrow. According to his idea, quaternions will be in place when we wish to have a convenient algorithm for the combination of rotations and dilatations; the true idea is that quaternions contains the elements of the algebra of space.

In investigating the fundamental principles of hyperboloidal trigonometry, the first problem is to find the general expression for a spherical versor, when reference is made to the axis of revolution.

Let OA (fig. 2) represent the axis of revolution, and let it be denoted by α. Any versor, POA, passing through the axis of revolution, may be denoted by $e^{b\sqrt{-1}\beta}$, where β denotes a unit axis perpendicular to α. Similarly AOQ, another versor, passing through the axis of revolution, may be denoted by $e^{c\sqrt{-1}\gamma}$, where γ denotes a unit axis perpendicular to α. The product versor POQ is circular, but it will not in general pass through OA; let it be denoted by $e^{a\sqrt{-1}\xi}$.

$$\text{Now } e^{a\sqrt{-1}\xi} = e^{b\sqrt{-1}\beta}e^{c\sqrt{-1}\gamma}$$

$$= (\mathrm{S}+\mathrm{V})(\mathrm{S}'+\mathrm{V}')$$

$$= \mathrm{SS}+\mathrm{SV}'+\mathrm{S}'\mathrm{V}+\mathrm{VV}'$$

$$= \cos b \cos c + \cos c \sin b\sqrt{-1}\beta + \cos b \sin c \sqrt{-1}\gamma + \sin b \sin c \sqrt{-1}\beta \sqrt{-1}\gamma;$$

$$= \cos b \cos c - \sin b \sin c \cos\beta\gamma$$

$$+ \sqrt{-1}\{\cos c \sin b{\cdot}\beta + \cos b \sin c.\gamma - \sin b \sin c \sin\beta\gamma.\overline{\beta\gamma}\}.$$

We observe that the directed sine may be broken up into two components—namely, $\cos c\ \sin b.\beta + \cos b\ \sin c\cdot\gamma$, which is perpendicular to the axis of revolution, and $-\sin b\ \sin c\ \sin\beta\gamma.\overline{\beta\gamma}$, which has the direction of the negative of the axis of revolution, for $\overline{\beta\gamma}$ is identical with a.

Draw OS to represent the first component $\cos c\ \sin b\cdot\beta$, OT to represent the second component $\cos b\ \sin c\cdot\gamma$, and OU to represent the third component $-\cos b\ \cos c\ \sin\beta\gamma\cdot a$ Draw OV, the resultant of the first two, and OR, the resultant of all three; then

$$\cos a = \cos b\ \cos c - \sin b\ \sin c\ \cos\beta\gamma$$

$$\text{and } \xi = \frac{\mathrm{OR}}{\sin a} = \frac{\cos c\ \sin b\cdot\beta + \cos b\ \sin c\cdot\gamma - \sin b\ \sin c\ \sin\beta\gamma\cdot a}{\sqrt{1-(\cos b\ \cos c\ - \sin b\ \sin c\ \cos\beta\gamma)^2}}.$$

The plane of OA and OV passes through OR, which is normal to the plane POQ; hence these planes cut orthogonally in a line OX, and the angle between OA and OX is equal to that between OV and OR, for OV is perpendicular to OA and OR to OX. Let ϑ denote the angle AOX; then

$$\sin\theta = \frac{\sin b\ \sin c\ \sin\beta\gamma}{\sqrt{1-(\cos b\ \cos c - \sin b\ \sin c\ \sin\beta\gamma)^2}}.$$

The figure (fig. 3) represents a section through the plane of OA and OV; MX represents $\sin\theta$. Hence the axis ξ can be put in the form $\cos\theta\cdot\epsilon - \sin\theta\cdot a$, where ϵ denotes a unit axis perpendicular to a. The unit axis ϵ may be expressed in terms of two axes j and k, forming an orthogonal system with the axis of revolution, which may be denoted by i. Hence a perfectly general expression for any spherical versor is $e^{a\sqrt{-1}\xi}$, where

$$\xi = \sqrt{-1}\{\cos\theta\cdot(\cos\phi\cdot j + \sin\phi\cdot k) - \sin\theta\cdot i\}.$$

We observe that if $e^{a\sqrt{-1}\xi}$ is an angle in the double sheet, $\sqrt{-1}\xi$ is a vector to the surface of the single sheet.

It is now easy to find the solution of the analogous problem, namely, the product of two diplanar hyperbolic versors when the plane of each passes through the axis of revolution.

The axis of the versor is perpendicular to the plane of the versor when the latter passes through the axis of revolution; and we shall assume that it is of unit length, an assumption which is afterwards

completely justified. Let the two versors POA and AOQ (fig. 4) be denoted by $e^{b\beta}$ and $e^{c\gamma}$, the axes β and γ being both perpendicular to the axis of revolution α, and of unit length.

Then
$$\begin{aligned} e^{b\beta}\,e^{c\gamma} &= (S+V)(S'+V') \\ &= SS' + S'V + SV' + VV' \\ &= \cosh b \cosh c + \cosh c \sinh b\cdot\beta + \cosh b \sinh c\cdot\gamma \\ &\qquad + \sinh b \sinh c\cdot\beta\gamma. \end{aligned}$$

Now
$$\begin{aligned} \beta\gamma &= \cos\beta\gamma + \sqrt{-1}\sin\beta\gamma\cdot\beta\gamma \\ &= \cos\beta\gamma + \sqrt{-1}\sin\beta\gamma\cdot\alpha. \end{aligned}$$

Hence
$$e^{b\beta}e^{c\gamma} = \cosh b\cosh c + \sinh b \sinh c \cos\beta\gamma$$
$$+ \cosh c \sinh b\cdot\beta + \cosh b\sinh c\cdot\gamma + \sqrt{-1}\sinh b\sinh c\sin\beta\gamma\cdot\alpha.$$

Hence
$$\cosh e^{b\beta}e^{c\gamma} = \cosh b\cosh c + \sinh b\sinh c\cos\beta\gamma$$

and
$$\begin{aligned} \mathrm{Sinh}\, e^{b\beta}e^{c\gamma} &= \cosh c\sinh b\cdot\beta + \cosh b\sinh c\cdot\gamma \\ &\qquad + \sqrt{-1}\sinh b\sinh c\sin\beta\gamma\cdot\alpha. \end{aligned}$$

The first and second components of the directed sinh (denoted by Sinh) are perpendicular to the axis of revolution, hence their resultant $\cosh c\sinh b\cdot\beta + \cosh b\sinh c\cdot\gamma$ is also perpendicular to the principal axis. Let it be represented by OV in the figure. The difficulty consists in finding the true direction of the third component $\sqrt{-1}\sinh b\sinh c\sin\beta\gamma\cdot\alpha$ on account of the presence of $\sqrt{-1}$. It will be found that $\sqrt{-1}$ has here nothing to do with the direction; and as the term is otherwise in the positive direction of α, we represent it by OU in the figure. In the case of the sphere OU is drawn in the direction opposite to α. Let OR be the resultant of OU and OV; it represents the directed Sinh both in magnitude and direction.

The square of the length of OR is

$$\begin{aligned} &\cosh^2 c\sinh^2 b + \cosh^2 b\sinh^2 c + 2\cosh c\cosh b\sinh c\sinh b\cos\beta\gamma \\ &\qquad + \sinh^2 b\sinh^2 c\sin^2\beta\gamma. \end{aligned}$$

But the square of the modulus of OR is the same with a negative sign before the last term; added to the square of $\cosh e^{b\beta}e^{c\gamma}$ it yields 1.

The directed sinh OR is not normal to the plane POQ; how is it related to that plane? If we draw OU′ = − OU and find OR′

the resultant, it is OR′ and not OR which is normal to the plane of OP and OQ. The expressions for the three vectors OR′, OP, OQ are

$$OR' = \cosh c \sinh b \cdot \beta + \cosh b \sinh c \cdot \gamma - \sinh b \sinh c \sin \beta\gamma \cdot a$$

$$OP = -\sinh b \frac{\cos \beta\gamma}{\sin \beta\gamma}.\beta + \sinh b \frac{1}{\sin \beta\gamma}.\gamma + \cosh b \cdot a$$

$$OQ = -\sinh c \frac{1}{\sin \beta\gamma}.\beta - \sinh c \frac{\cos \beta\gamma}{\sin \beta\gamma}.\gamma + \cosh c \cdot \gamma$$

from which it follows that S(OR′)(OP)=0 and S(OR′)(OQ)=0. Hence OR′ is normal to the plane of POQ. How is the direction of OR related to that plane? The plane of OA and OV (fig. 5) cuts the equilateral hyperboloid in an equilateral hyperbola; and as it passes through the normal OR′, it must cut the plane POQ orthogonally.

Let OX be the line of intersection. Draw XM perpendicular to OA, draw XD a tangent to the equilateral hyperbola at X (fig. 5), and XA′ parallel to OA. Let θ denote the hyperbolic angle AOX. As OR′ is normal to the plane POQ, it is perpendicular to OX; but OV is perpendicular to OA, therefore the angle AOX is equal to the angle VOR′. Now the angle AOR is the complement of ROV, and A′XD the complement of AOX; therefore the line OR is parallel to the tangent XD. Thus the direction of the directed sinh is that of the conjugate axis to the plane of OP and OQ. This idea of *conjugate* instead of *normal* also applies to the spherical case, from which it follows that $ij = \sqrt{-1}k$ means that k is the axis conjugate to i and j.

$$\text{Now } \sinh \theta = \frac{MX}{OA} = \frac{VR}{\sqrt{OV^2 - VR^2}}$$

$$= \frac{\sinh b \sinh c \sin \beta\gamma}{\sqrt{(\cosh b \cosh c + \sinh b \sinh c \cos \beta\gamma)^2 - 1}}$$

The above analysis shows that the product versor POQ may be specified by the following three elements:—*First*, ϵ, a unit axis drawn perpendicular to OA in the plane of OA and the normal to the plane POQ; *second*, θ, the hyperbolic angle determined by OA and OX, which is drawn at right angles to the normal in the plane of OA and the normal; *third*, a, the angle of the hyperbolic sector

OPXQ, which is a sector of the hyperbola having OX for semi-major axis, and for semi-minor axis OB which is equal to OA and perpendicular to OA and OV. This hyperbola is not an equilateral hyperbola; PXQ is the curve of intersection of the hyperboloid with a plane through the points O, P, Q. An angle of this hyperbola is specified by the ratio of the sector to half of the rectangle formed by OX and OB. Thus a is the ratio of the sector POQ to half of the rectangle formed by OX and OB.

Hence the product versor may be expressed by means of a hyberbolic angle a and a hyberbolic axis of the form

$$\cosh\theta\cdot\epsilon + \sqrt{-1}\sinh\theta\cdot a,$$

where, as before, ϵ denotes a unit axis normal to a, the axis of revolution. Let ξ denote the above axis; the actual components from which it is constructed are $\cosh\theta\cdot\epsilon$ and $\sinh\theta\cdot a$ It is not of unit length, but it has a unit modulus. The former is $\sqrt{\cosh^2\theta + \sinh^2\theta}$, the latter is $\sqrt{\cosh^2\theta - \sinh^2\theta}$.

Hence the product versor may be expressed by

$$e^{a\xi} = e^{a(\cosh\theta\cdot\epsilon + \sinh\theta\cdot a)}.$$

And to determine these quantities we have the three analogous equations

$$\cosh a = \cosh b\cosh c + \sinh b\sinh c\cos\beta\gamma \quad (1)$$

$$\cosh\theta = \frac{\sinh b\sinh c\sin\beta\gamma}{\sinh a}$$

$$\epsilon = \frac{\cosh c\sinh b\cdot\beta + \cosh b\sinh c\cdot\gamma}{\sinh a\sinh\theta.}$$

As ϵ is of unit length, it may be expressed as $\cos\phi\cdot j + \sin\phi\cdot k$, and if i denotes the axis of revolution

$$\xi = \cosh\theta(\cos\phi\cdot j + \sin\phi\cdot k) + \sqrt{-1}\sinh\theta\cdot i.$$

The axis ξ is evidently a vector to a point in the conjugate hyperboloid of one sheet.

In the above investigation it is assumed that the magnitude of the perpendicular component of the Sinh is necessarily greater than the component parallel to the axis of revolution. This means that

$$\cosh^2 c\sinh^2 b + \cosh^2 b\sinh^2 c + 2\cosh b\cosh c\sinh b\sinh c\cos\beta\gamma$$
$$> \sinh^2 b\sinh^2 c\sin^2\beta\gamma.$$

Let $\sin\beta\gamma = 1$, $\cos\beta\gamma = 0$; then each of the two terms on the left is greater than the term on the right of the inequality. Let

sin $\beta\gamma = 0$ and $\cos\beta\gamma = -1$, then the above expression reduces to the well known inequality $a^2 + b^2 > 2\,ab$. Hence the terms on the left are always greater than the term on the right.

In the case when the two versors are equal, we can verify that it is the line of intersection of the central plane with the equilateral hyperboloid which is indicated by the product of the versors.

As the two versors are equal they might be denoted by $e^{b\beta}$ and $e^{b\gamma}$. Let $\cosh b = x$, $\sinh b = y$. Then according to the theorem

$$e^{b\beta}\,e^{b\gamma} = x^2 + y^2\cos\beta\gamma + xy\;(\beta + \gamma) + \sqrt{-1}y^2\sin\beta\gamma\cdot a$$

As (fig. 6), OB the semi-transverse axis of the hyperbola PXQ is 1, NQ represents the sinh of half of the product angle. Now by the geometry of the construction

$$\frac{\mathrm{NQ}}{\mathrm{OB}} = \tfrac{1}{2}\sqrt{2y^2 + 2y^2\cos\beta\gamma}$$

$$= \frac{y}{\sqrt{2}}\sqrt{1 + \cos\beta\gamma}.$$

Again $\dfrac{\mathrm{ON}}{\mathrm{OX}} = \dfrac{x}{\cosh\theta}$

$$= \frac{x\sqrt{(x^2 + y^2\cos\beta\gamma)^2 - 1}}{\sqrt{x^2\,y^2\,2\,(1 + \cos\beta\gamma)}}$$

$$= \sqrt{1 + \frac{y^2}{2}(1 + \cos\beta\gamma)}.$$

Now $\cosh 2\mathrm{XOQ} = (\cosh\mathrm{XOQ})^2 + (\sinh\mathrm{XOQ})^2$

$$= \left(\frac{\mathrm{NQ}}{\mathrm{OB}}\right)^2 + \left(\frac{\mathrm{ON}}{\mathrm{OX}}\right)^2$$

$$= \frac{y^2}{2}(1 + \cos\beta\gamma) + 1 + \frac{y^2}{2}(1 + \cos\beta\gamma)$$

$$= 1 + y^2 + y^2\cos\beta\gamma$$
$$= x^2 + y^2\cos\beta\gamma$$

which agrees with the above theorem.

We have seen that the general spherical versor is denoted by $e^{a\sqrt{-1}\xi}$, where

$$\xi = -\sin\theta\cdot a + \cos\theta\cdot\epsilon,$$

a denoting the axis of revolution and ϵ an axis in the perpendicular plane. Similarly a general versor for the equilateral hyperboloid of two sheets is denoted by $e^{a\xi}$, where

$$\xi = \sqrt{-1}\sinh\theta\cdot a + \cosh\theta\cdot\epsilon,$$

α and ϵ denoting the same kind of axes as before. This leads us to the consideration of hyperboloidal axes. Let ξ_1 denote a radius to the double sheet (fig. 7);

$$\xi_1 = \cosh\theta\cdot\alpha + \sqrt{-1}\sinh\theta\cdot\epsilon.$$

The length of ξ_1 is

$$\sqrt{\cosh^2\theta + \sinh^2\theta}$$

but its modulus is $\sqrt{\cosh^2\theta - \sinh^2\theta}$, which is 1. Let ξ_2 denote a radius to the single sheet;

$$\xi_2 = \sqrt{-1}\sinh\theta\cdot\alpha + \cosh\theta\cdot\epsilon.$$

The corresponding axes for the unit sphere are

$$\xi_1 = \cos\theta\cdot\alpha + \sin\theta\cdot\epsilon$$

$$\text{and } \xi_2 = -\sin\theta\cdot\alpha + \cos\theta\cdot\epsilon.$$

Just as a spherical vector is expressed by $r\sqrt{-1}\xi$, so a hyperboloidal vector is expressed by $r\xi$, where r denotes the modulus and ξ the axis. The principal difference is that in the case of the sphere ξ is of constant length, whereas in the case of the hyperboloid the length of the axis depends on its position relative to the axis of revolution.

Consider now a general triangle on the hyperboloid of two sheets (fig. 8). Let the axes to the three points be denoted by

$$\xi = \cosh\theta\cdot\alpha + \sqrt{-1}\sinh\theta\cdot\beta$$

$$\eta = \cosh\theta'\cdot\alpha + \sqrt{-1}\sinh\theta'\cdot\gamma$$

$$\zeta = \cosh\theta''\cdot\alpha + \sqrt{-1}\sinh\theta''\cdot\delta.$$

Then $\xi\eta = \cosh\theta\cosh\theta' - \sinh\theta\sinh\theta'\cos\beta\gamma$ (1)

$-\cosh\theta\sinh\theta'\cdot\overline{\alpha\gamma} - \sinh\theta\cosh\theta'\cdot\overline{\beta\alpha}$ (2)

$-\sqrt{-1}\sinh\theta\sinh\theta'\sin\beta\gamma\cdot\alpha$ (3)

Hence $\cosh\xi\eta = (1)$

and $\operatorname{Sinh}\xi\eta = (2) + (3)$.

We have proved that the length of (3) is always less than the length of (2); hence $\overline{\xi\eta}$ has the form

$$\sinh\phi\cdot\alpha + \sqrt{-1}\cosh\phi\cdot\epsilon.$$

And the same is true for $\eta\zeta$ and $\zeta\xi$. The central section is always hyperbolic.

Now $\xi\zeta = (\xi\eta)(\eta\zeta)$.

Therefore $\cosh\xi\zeta = \cosh\xi\eta\cosh\eta\zeta + \cosh(\operatorname{Sinh}\xi\eta\operatorname{Sinh}\eta\zeta)$ and $\operatorname{Sinh}\xi\zeta = \cos\eta\zeta\operatorname{Sinh}\xi\eta + \cosh\xi\eta\operatorname{Sinh}\zeta\eta$

$$+\operatorname{Sinh}\{\operatorname{Sinh}\xi\eta\operatorname{Sinh}\eta\zeta\}.$$

Consider now a general triangle on the hyberboloid of one sheet (fig. 9).

Let the three axes be

$$\xi = \cosh\theta\cdot\beta + \sqrt{-1}\sinh\theta\cdot a$$
$$\eta = \cosh\theta'\cdot\gamma + \sqrt{-1}\sinh\theta'\cdot a$$
$$\zeta = \cosh\theta''\cdot\delta + \sqrt{-1}\sinh\theta''\cdot a.$$

Then $\xi\eta = \cosh\theta\cosh\theta'\cos\beta\gamma - \sinh\theta\sinh\theta'$ (1)

$- \cosh\theta\sinh\theta'\cdot\overline{\beta a} - \cosh\theta'\sinh\theta\cdot\overline{a\gamma}$ (2)

$+ \sqrt{-1}\cosh\theta\cosh\theta'\sin\beta\gamma\cdot a$ (3)

In this case the length of the normal part of the Sinh may be greater than, equal to, or less than the length of the components along the axis of revolution. For we have to compare—

$\cosh^2\theta\sinh^2\theta' + \cosh^2\theta'\sinh^2\theta - 2\cosh\theta\cosh\theta'\sinh\theta\sinh\theta'$ $\cos\beta\gamma$ with $\cosh^2\theta\cosh^2\theta'\sin^2\beta\gamma$. Let $\sin\beta\gamma = 0$, $\cos\beta\gamma = -1$; then the former term is the greater. Let $\cos\beta\gamma = 0$, $\sin\beta\gamma = 1$; then the former term is the less. And the terms may be equal. In the former case the axis of $\xi\eta$ has the form

$$\cosh\phi\cdot\epsilon + \sqrt{-1}\sinh\phi\cdot a$$

and the section is hyperbolic. In the latter case the axis of $\xi\eta$ has the form

$$\sqrt{-1}\,\{\cosh\theta\cosh\theta'\sin\beta\gamma\cdot a + \sqrt{-1}\,(\cosh\theta\sinh\theta'\cdot\overline{\beta a} + \cosh\theta'\sinh\theta\cdot\overline{a\gamma})\}.$$

The axis inside the brackets denotes an axis of the equilateral hyperboloid of two sheets, and the section is elliptic.

As before

$$\xi\zeta = (\xi\eta)\,(\eta\zeta)$$

therefore $\cosh\xi\zeta = \cosh\xi\eta\cosh\eta\zeta + \cosh\{\mathrm{Sinh}\,\xi\eta\ \mathrm{Sinh}\,\eta\zeta\}$

and

$\mathrm{Sinh}\,\xi\eta = \cosh\eta\zeta\ \mathrm{Sinh}\,\xi\eta + \cosh\xi\eta\ \mathrm{Sinh}\,\eta\zeta + \mathrm{Sinh}\,\{\mathrm{Sinh}\,\xi\eta\ \mathrm{Sinh}\,\eta\zeta\}.$

HYPERBOLIC QUATERNIONS.

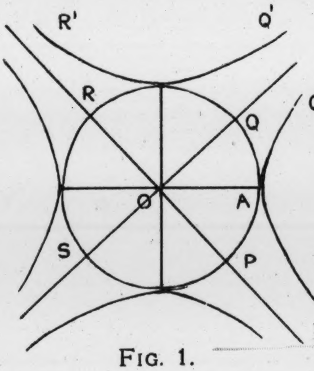


FIG. 1.

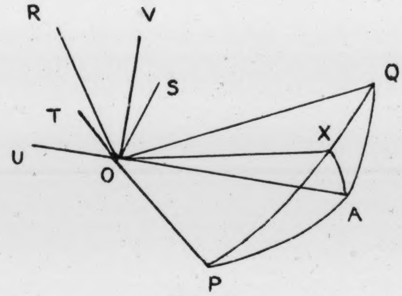


FIG. 2.

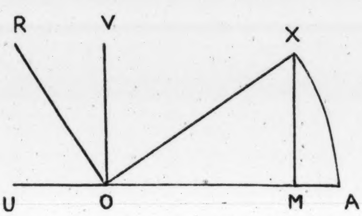


FIG. 3.

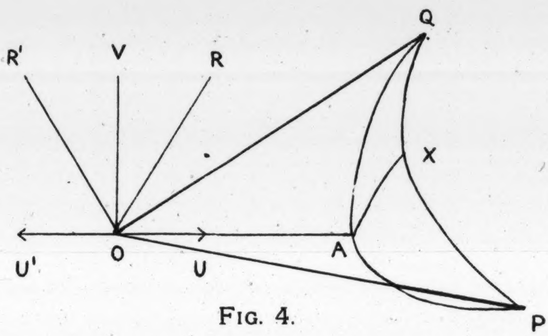


FIG. 4.

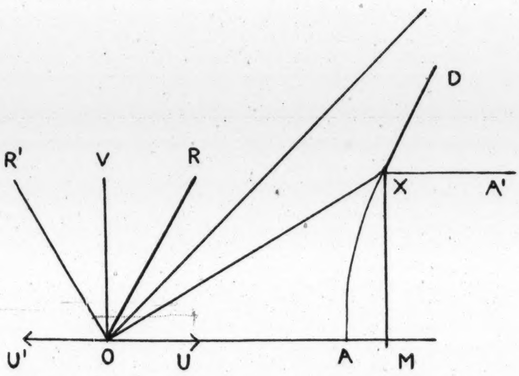


FIG. 5.

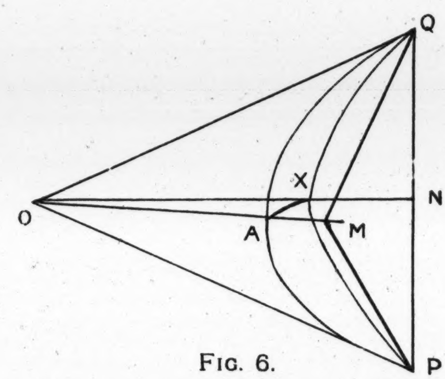


FIG. 6.

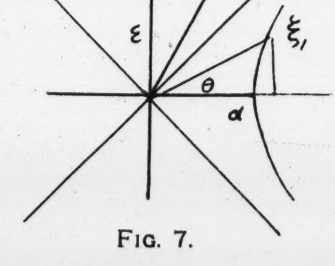


FIG. 7.

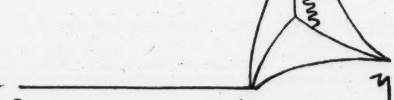


FIG. 8.

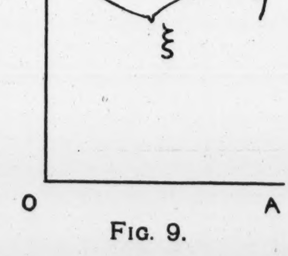


FIG. 9.

A. RITCHIE & SON EDINR.

The Theory of Skew Determinants and Pfaffians in the Historical Order of its Development up to 1857. By Thomas Muir, LL.D.

(Read July 16, 1900.)

Sets of equations of the form

$$\left.\begin{array}{rrrrrrl}
 & a_{12}x_2 & + \; a_{13}x_3 & + \; a_{14}x_4 & + \; \ldots\ldots & + \, a_{1n}x_n & = \xi_1 \\
-a_{12}x_1 & & + \; a_{23}x_3 & + \; a_{24}x_4 & + \; \ldots\ldots & + \, a_{2n}x_n & = \xi_2 \\
-a_{13}x_1 & - \; a_{23}x_2 & & + \; a_{34}x_4 & + \; \ldots\ldots & + \, a_{3n}x_n & = \xi_3 \\
-a_{14}x_1 & - \; a_{24}x_2 & - \; a_{34}x_3 & & + \; \ldots\ldots & + \, a_{4n}x_n & = \xi_4 \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots \\
-a_{1n}x_1 & - \; a_{2n}x_2 & - \; a_{3n}x_3 & - \; a_{4n}x_4 & - \; \ldots\ldots & & = \xi_n
\end{array}\right\},$$

where the coefficient of x_r in the s^{th} equation differs only in sign from the coefficient of x_s in the r^{th} equation, had often made their appearance in analytical investigations before the determinant of such a set came to be considered. An instance is to be found in a memoir of Poisson's, read before the Institute in October 1809, and printed in the *Journal de l'École Polytechnique*, viii., pp. 266 — 344*; and similar instances of an earlier date in writings of Lagrange and Laplace therein referred to. The mathematician who first referred definitely to the determinant appears to have been Jacobi.

JACOBI (1827).

[Ueber die Pfaffsche Methode, eine gewöhnliche lineare Differential-gleichung zwischen $2n$ Variabeln durch ein System von n Gleichungen zu integriren. *Crelle's Journ.*, ii. pp. 347–357.]

An essential part of Pfaff's method is the solution of a set of equations which Jacobi writes in the form

* See especially p. 288.

$$\left.\begin{array}{rcl}
NX\partial x &=& \ast \quad + (0,1)\partial x_1 + (0,2)\partial x_2 + \ldots + (0,p)\partial x_p \\
NX_1\partial x &=& (1,0)\partial x \quad \ast \quad + (1,2)\partial x_2 + \ldots + (1,p)\partial x_p \\
NX_2\partial x &=& (2,0)\partial x + (2,1)\partial x_1 + \quad \ast \quad + \ldots + (2,p)\partial x_p \\
\ldots & & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
NX_p\partial x &=& (p,0)\partial x + (p,1)\partial x_1 + (p,2)\partial x_2 + \ldots + \quad \ast
\end{array}\right\},$$

where $(0,0) = -(1,0)$ and generally $(a,\beta) + (\beta,a) = 0$. This form of his own he frankly characterises as "elegant and completely symmetrical"; but the same description would apply equally appropriately to the solution which he gives. Unfortunately, the method by which the latter was obtained is not indicated, and we can only hazard a guess in regard to it. The balance of probability would seem to be in favour of the method of devising a set of multipliers which, when applied to the given equations, would after the performance of addition bring about the elimination of all the unknowns except one. In the case of four equations this would not be at all difficult. For example, if we wish to eliminate x_2, x_3, x_4 from the equations

$$\left.\begin{array}{rcccl}
. & ax_2 & + bx_3 & + cx_4 & = \xi_1 \\
-ax_1 & . & + dx_3 & + ex_4 & = \xi_2 \\
-bx_1 & - dx_2 & . & + fx_4 & = \xi_3 \\
-cx_1 & - ex_2 & - fx_3 & . & = \xi_4
\end{array}\right\},$$

the multipliers are readily seen to be

$$0,\ f,\ -e,\ d,$$

so that after multiplication and addition there results

$$(-af + be - cd)x_1 = f\xi_2 - e\xi_3 + dx_4 .$$

Similarly by using the multipliers $-f$, 0, c, $-b$ we find

$$(-af + be - cd)x_2 = -f\xi_1 + c\xi_3 - b\xi_4 ;$$

and the other two are

$$(-af + be - cd)x_3 = e\xi_1 - c\xi_2 + a\xi_4 ,$$
$$(-af + be - cd)x_4 = -d\xi_1 + b\xi_2 - a\xi_3 .$$

Jacobi's corresponding result is to the effect that the numerators of the values of the four unknowns are

$$\begin{array}{l}
N\partial x\{\ \ast \quad + (2,3)X_1 + (3,1)X_2 + (1,2)X_3\}, \\
N\partial x\{(3,2)X + \quad \ast \quad + (0,3)X_2 + (2,0)X_3\}, \\
N\partial x\{(1,3)X + (3,0)X_1 + \quad \ast \quad + (0,1)X_3\}, \\
N\partial x\{(2,1)X + (0,2)X_1 + (1,0)X_2 + \quad \ast \quad \},
\end{array}$$

and the common denominator

$$(0,1)(3,2) + (0,3)(2,1) + (0,2)(1,3),$$

or, as he thereafter writes it

$$(0,1,3,2).$$

When the similar set of six equations came to be dealt with, the devising of the multipliers requisite for elimination would necessarily be harder; but keeping in view the analogous mode of arriving at the solution of

$$\left.\begin{array}{l} a_1x_1 + a_2x_2 = \xi_1 \\ b_1x_1 + b_2x_2 = \xi_2 \end{array}\right\}$$

and then proceeding to the solution of

$$\left.\begin{array}{l} a_1x_1 + a_2x_2 + a_3x_3 = \xi_1 \\ b_1x_1 + b_2x_2 + b_3x_3 = \xi_2 \\ c_1x_1 + c_2x_2 + c_3x_3 = \xi_3 \end{array}\right\},$$

where, it will be remembered, the multipliers requisite for elimination are of the same form as the common denominator of the values of the unknowns in the preceding case, Jacobi would have little real difficulty in finding that corresponding to the four multipliers requisite for eliminating $\partial x_1, \partial x_2, \partial x_3$ in his first case, viz.,—

$$0,\ (2,3),\ (3,1),\ (1,2)$$

he would now require to have the six multipliers

$$0,\ (2345),\ (3451),\ (4512),\ (5123),\ (1234).$$

As a matter of fact, he gives for the numerator of the first unknown

$$\mathrm{N}\partial x\{\ *\ + (2345)\mathrm{X}_1 + (3451)\mathrm{X}_2 + (4512)\mathrm{X}_3 + (5123)\mathrm{X}_4 + (1234)\mathrm{X}_5\},$$

the others being

$$\mathrm{N}\partial x\{(3245)\mathrm{X} +\ *\ + (4350)\mathrm{X}_2 + (5402)\mathrm{X}_3 + (0523)\mathrm{X}_4 + (2034)\mathrm{X}_5\}$$

. . . .

The common denominator is not mentioned; we should have expected him to say that it was

$$(10)(2345) + (20)(3451) + (30)(4512) + (40)(5123) + (50)(1234)$$

or $$-(012345).$$

It is then pointed out that when the first coefficient has been got in one of the numerators, the others are arrived at by circular permutation, the elements permuted being 12345 in the case of the first numerator, 02345 in the case of the second, 01345 in the case of the third, and so on; also that the first coefficient in one line is got from the last in the preceding line by changing 012345 into 123450 and then transposing the first two elements; and that these laws hold generally.

A general mode of finding the ordinary expression for the new functions here introduced and symbolized by

$$(1234), (123456), \ldots .$$

is next explained. It is first stated that the number of terms represented by

$$(2,3,4, \ldots, p)$$

where p is necessarily an odd integer is

$$1.3.5. \ldots . (p-2),$$

and that one of them is

$$(23).(45).(67) \ldots . (p-1, p).$$

We are then told to permute cyclically the last $p-2$ elements $3,4,5, \ldots, p$, and we shall obtain from this $p-2$ terms in all; thereafter to permute cyclically the last $p-4$ elements $5,6,7, \ldots p$ in each of the $p-2$ terms just obtained, and so on. For example,

$$\begin{aligned}(234567) = \;& (23)(45)(67) + (23)(46)(75) + (23)(47)(56) \\ & + (24)(56)(73) + (24)(57)(36) + (24)(53)(67) \\ & + (25)(67)(34) + (25)(63)(47) + (25)(64)(73) \\ & + (26)(73)(45) + (26)(74)(53) + (26)(75)(34) \\ & + (27)(34)(56) + (27)(35)(64) + (27)(36)(45).\end{aligned}$$

It is important to note in conclusion, that the case of an *odd* number of equations is not neglected by Jacobi, a proof being given by him that in that case the determinant of the system vanishes. In his own words—which are interesting in view of what has been said elsewhere regarding the evidence which the paper affords of the progress made by him in the study of determinants—

"Nun bleibt nach dem bekannten Algorithmus, nach welchem die Determinante gebildet wird, diese unverändert,

wenn man die Horizontalreihen und Verticalreihen der Coefficienten mit einander vertauscht. Für unsern besondern Fall nun wird, wenn wir die Determinante mit Δ bezeichnen, hieraus folgen: $\Delta = (-1)^{p+1}\,\Delta$, da jedes Glied der Determinante ein Product aus $p+1$ Coefficienten ist, von denen jeder durch Vertauschung der Horizontal- und Verticalreihen sich in sein Negatives verwandelt. Diese Gleichung $\Delta = (-1)^{p+1}\Delta$ aber kann nur bestehen, wenn $p+1$ eine gerade Zahl ist, wofern nicht $\Delta = 0$ sein soll."

Thus, besides being the originator of the functions which came long afterwards to be known and are still known as 'Pfaffians,' Jacobi was the first to discover and prove the now familiar-worded theorem "*A zero-axial skew determinant of odd order vanishes.*"

JACOBI (1845).

[Theoria novi multiplicatoris systemati æquationum differentialium vulgarium applicandi. *Crelle's Journ.*, xxvii. pp. 199–268, xxix. pp. 213–279, 333–376.]

As is well known, this long and exhaustive memoir of Jacobi's is broken up into three chapters,—the first giving the definition and various properties of the new multiplier, the second explaining the application of it to the integration of differential equations, and the third illustrating this application by means of particular differential equations of historical interest. One of the latter is the equation associated then, and still more since, with the name of Pfaff, the discussion of it occupying §§ 20, 21 on pp. 236–253 of Vol. xxix. We are thus prepared to find the function, defined by Jacobi eighteen years before, again referred to.

The old definition of the function, which he here denotes by R, is practically repeated, the initial and originating term being now of the form $a_{12}a_{34}\,.\,.\,.\,a_{2m-1,2m}$: and then he makes the pregnant general remark that the properties of R are analogous to those of determinants. Prominence is given to the theorem regarding the effect of interchanging two indices. This is followed by the twin pair of identities

$$R = a_{1,s}\frac{\partial R}{\partial a_{1,s}} + a_{2,s}\frac{\partial R}{\partial a_{2,s}} + \dots + a_{2m,s}\frac{\partial R}{\partial a_{2m,s}},$$

$$0 = a_{1,s}\frac{\partial R}{\partial a_{1r}} + a_{2s}\frac{\partial R}{\partial a_{2r}} + \dots + a_{2m,s}\frac{\partial R}{\partial a_{2m,r}},$$

in the latter of which s differs from r, and the term $a_{rs}\dfrac{\partial R}{\partial a_{rr}}$ is awanting; and finally, it is pointed out that the differential-quotient of R with respect to one or more elements are functions of the same kind as the original, and, probably as a consequence, that certain second differential-quotients are identical. No proofs are given; indeed, the statements themselves are in the most concise form possible, the whole passage being as follows:—

> "Designantibus i, i', i'', etc., indices inter se diversos, si sumuntur differentialia partialia
>
> $$\frac{\partial R}{\partial a_{i,i'}}, \quad \frac{\partial^2 R}{\partial a_{i,i'}\,\partial a_{i'',i'''}}, \quad \dots$$
>
> ea erunt aggregata ad instar aggregati R formata, respective reiectis Coëfficientium binis, quatuor, . . . seriebus cum horizontalibus tum verticalibus, eritque
>
> $$\frac{\partial^2 R}{\partial a_{ii'}\,\partial a_{i''i'''}} = \frac{\partial^2 R}{\partial a_{i,i''}\,\partial a_{i'''i'}} = \frac{\partial^2 R}{\partial a_{i,i'''}\,\partial a_{i',i''}}."$$

It should be carefully noted that both in this paper and in the preceding, Jacobi views the new functions as separate from and independent of determinants, and not at all in the light in which, at a later time, they came to be looked upon—viz., as a subsidiary function arising out of the study of a particular kind of determinant with which it had a definite quantitative relation.

CAYLEY (1846).

[Sur quelques propriétés des déterminants gauches. *Crelle's Journ.*, xxxii. pp. 119–123; or *Collected Math. Papers*, i. pp. 332–336.]

This paper, with its author's usual directness, starts at once with a definition, the first words being—

"Je donne le nom de *déterminant gauche* à un déter-

minant formé par un système de quantités $\lambda_{r,s}$ qui satisfont aux conditions

$$\lambda_{r,s} = -\lambda_{s,r} \qquad (r \neq s).$$

J'appelle aussi un tel système, *système gauche*."

So far as can be ascertained, the English equivalent '*skew*,' although it probably was the first of the two in order of thought, did not appear in print until a few years later.

As has been pointed out elsewhere, the title of the paper is quite misleading, the real subject being *the construction of a linear substitution for the transformation of* $x_1^2 + x_2^2 + x_3^2 + \ldots$ *into* $\xi_1^2 + \xi_2^2 + \xi_3^2 + \ldots$ All that can be said in defence of the inaccuracy is that skew determinants are made use of in obtaining the desired substitution. The proper place for giving an account of the contents of the paper is thus under the heading of '*orthogonants*,' if we may so name the *determinants of an orthogonal substitution.*

CAYLEY (1847).

[Sur les déterminants gauches. *Crelle's Journ.*, xxxviii. pp. 93–96; or *Collected Math. Papers*, i. pp. 410–413.]

Here the title and contents agree. At the outset the former definition is repeated, and then for a particular kind of skew determinant, viz., those in which the condition

$$\lambda_{r,s} = -\lambda_{s,r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

is to hold even in the case where s and r are equal, "ou pour lesquels on a

$$\lambda_{r,s} = -\lambda_{s,r} \quad (r \neq s), \qquad \lambda_{r,r} = 0\text{''}, \quad . \quad . \quad . \quad . \qquad (2)$$

the name 'skew symmetric' (gauche et symétrique") is set apart. The reason for this is evident on the statement of the first theorem, which is to the effect that any skew determinant is expressible in terms of skew symmetric determinants and those elements of the original determinant which are not included in the latter. "En effet," he explains,

"soit Ω le déterminant gauche dont il s'agit, cette fonction peut être présentée sous la forme

$$\Omega = \Omega_0 + \Omega_1\lambda_{1,1} + \Omega_2\lambda_{2,2} + \ldots + \Omega_{12}\lambda_{11}\lambda_{22} + \ldots$$

où Ω_0 est ce que devient Ω si $\lambda_{11}, \lambda_{22}, \ldots$ sont réduits à zéro, Ω_1 est ce que devient le coefficient de λ_{11} sous la même condition, et ainsi de suite; c'est à dire, Ω_0 est le déterminant formé par les quantités $\lambda_{r,s}$ en supposant que ces quantités satisfassent aux conditions (2) et en donnant à r,s les valeurs $1, 2, 3, \ldots, n$; Ω_1 est le déterminant formé pareillement en donnant à r,s les valeurs $2, 3, \ldots, n$; Ω_2 s'obtient en donnant à r,s les valeurs $1, 3, \ldots, n$; et ainsi de suite; cela est aisé de voir si l'on range les quantités $\lambda_{r,s}$ en forme de carré."

At this point a digression is made in order to establish a theorem regarding skew determinants of odd order, and another regarding skew determinants of even order, and thus be enabled to make certain substitutions for the Ω's in the development here announced. Further, as the said substitutions for the Ω's of even order involve the functions dealt with by Jacobi in his paper on the "Pfaffsche Methode,"—functions which Cayley here calls "les fonctions de M. Jacobi," but which at a later date he designated "*Pfaffians*,"—the digression is lengthened by having prefixed to it an account of these functions.

So curious is this account and so likely to be misrepresented by condensation, that the best way of treating it is to reproduce it in the original words.* It stands thus:—

"On obtient ces fonctions (dont je reprends ici la théorie) par les propriétés générales d'un déterminant défini. Car en exprimant par $(1, 2, \ldots, n)$ une fonction quelconque dans laquelle entrent les nombres symboliques $1, 2, \ldots, n$, et par $\pm$ le signe correspondant à une permutation quelconque de ces nombres, la fonction

$$\sum \pm (1\ 2\ \ldots\ n)$$

où $\sum$ désigne la somme de tous les termes qu'on obtient en

* The paper, as it appears in *Crelle's Journal*, is disfigured by misprints, which have not been fully corrected in the *Collected Math. Papers*.

permutant ces nombres d'une manière quelconque) est ce qu'on nomme *Déterminant*. On pourrait encore généraliser cette définition en admettant plusieurs systèmes de nombres $1, 2 \ldots, n;\ 1', 2' \ldots, n';\ \ldots$ qui alors devroient être permutés independamment les uns des autres; on obtiendrait de cette manière une infinité d'autres fonctions, mentionnées (T. xxx. p. 7). Dans le cas des déterminants ordinaires, auquel je ne m'arrêterai pas ici, on aura $(1, 2 \ldots n) = \lambda_{\alpha,1}\lambda_{\beta,2} \ldots \lambda_{\kappa,n}$. Pour les cas des fonctions dont il s'agit (les fonctions de M. Jacobi), on supposera *n pair*, et l'on écrira

$$(1\ 2\ \ldots n) = \lambda_{1,2}\lambda_{3,4} \ldots \lambda_{n-1,n},$$

où $\lambda_{r,s}$ sont des quantités quelconques qui satisfont aux équations (1). La fonction sera composée d'un nombre $1.2 \ldots n$ de termes; mais parmi eux il n'y aura que $1.3 \ldots (n-1)$ termes différents qui se trouveront répétés $2^{\frac{1}{2}n}(1.2 \ldots \frac{1}{2}n)$ fois, et qu'on obtiendra en permutant cycliquement d'abord les $n-1$ derniers nombres, puis les $n-3$ derniers nombres de chaque permutation, et ainsi de suite; le signe étant toujours $+$. Il pourra être démontré, comme pour les déterminants, que ces fonctions changent de signe en permutant deux quelconques des nombres symboliques, et qu'elles s'evanouissent si deux de ces nombres deviennent identiques. De plus, en exprimant par $[12 \ldots n]$ la fonction dont il s'agit, la règle qui vient d'être énoncé, donnera pour la formation de ces fonctions:

$$[1\,2 \ldots n] = \lambda_{12}[3\,4 \ldots n] + \lambda_{13}[4 \ldots n, 2] + \ldots\ldots + \lambda_{1n}[2\,3 \ldots n-1].$$

Dismissing, as not of present interest, the sentence regarding the generalisation obtained by admitting more than one system of symbolic numbers, we note first of all the peculiar general use of $(1\ 2 \ldots n)$ for any function the expression of which involves* as suffixes or otherwise the numbers $1, 2, 3, \ldots, n$. Then we are struck with the fact that the use of this along with $\Sigma\pm$ gives a

* Apparently it is meant to be implied that each of the numbers occurs only once in the expression.

notation for a genus of functions of which determinants, as understood up to the date of the paper, formed a species: thus

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2$$

is the case of $\Sigma \pm (123)$ where $(123) = a_1b_2c_3$. In the third place we are surprised to find that Cayley seems to propose to extend the meaning of the word *determinant* by transferring the name of the species to the genus, and to call by the name of "ordinary determinants" the functions formerly known as "determinants" merely.

All this is in itself comparatively unimportant, serving perhaps only to recall to us Cauchy's famous paper of 1812, where we have K, the originating term of an alternating function to compare and contrast with Cayley's $(12 \ldots n)$, and 'alternating function' to compare and contrast with Cayley's extended meaning of 'determinant.' But what follows by way of second example is very noteworthy, because the originating term taken, viz., $\lambda_{12}\lambda_{34} \ldots \lambda_{n-1,n}$ is one that could not possibly have been used by Cauchy, with whom Σ denoted an operation of a much less simple character than permutation of the integers $1, 2, \ldots, n$. Unfortunately the example is not fully exploited.* We are only told that in a certain special

* Supplying this defect we see that in strict accordance with Cayley's definition

$$\begin{aligned}
\Sigma \pm 12{\cdot}34 = \quad & 12{\cdot}34 + 31{\cdot}24 \\
& - 12{\cdot}43 - 31{\cdot}42 \\
& - 13{\cdot}24 - 32{\cdot}14 \\
& + 13{\cdot}42 + 32{\cdot}41 \\
& + 14{\cdot}23 + 34{\cdot}12 \\
& - 14{\cdot}32 - 34{\cdot}21 \\
& - 21{\cdot}34 - 41{\cdot}23 \\
& + 21{\cdot}43 + 41{\cdot}32 \\
& + 23{\cdot}14 + 42{\cdot}13 \\
& - 23{\cdot}41 - 42{\cdot}31 \\
& - 24{\cdot}13 - 43{\cdot}12 \\
& + 24{\cdot}31 + 43{\cdot}21, \\
= 2\{ & 12{\cdot}34 - 12{\cdot}43 - 13{\cdot}24 + 13{\cdot}42 \\
& + 14{\cdot}23 - 14{\cdot}32 - 21{\cdot}34 + 21{\cdot}43 \\
& - 23{\cdot}41 + 24{\cdot}31 - 31{\cdot}42 + 32{\cdot}41\},
\end{aligned}$$

—a function of twelve variables which is not a determinant in the acceptation either of the present time or of the time preceding Cayley.

case, viz., where the elements are such that rs is always equal to $-sr$, there are only $1 . 3 . 5 \ldots (2n-1)$ different terms in

$$\sum \pm \lambda_{12}\lambda_{34} \ldots \lambda_{2n-1,2n} ;$$

that the aggregate of these is also got without repetition in a particular way already announced by Jacobi; and that it is this aliquot part of $\Sigma \pm \lambda_{12}\lambda_{34} \ldots \lambda_{2n-1,n}$ which constitutes 'la fonction de M. Jacobi.' Jacobi's theorem regarding the effect, on the function, of interchanging two indices, is then restated; and a step further is taken in affirming that the function vanishes when two indices are equal. Finally, another law of formation—the recurring law—is given in the form

$$[12 \ldots 2n] = 12[345 \ldots 2n] + 13[45 \ldots 2n,2] + 14[5 \ldots 2n,2,3] + \ldots$$

which, of course, is in substance not different from Jacobi's

$$\mathrm{R} = a_{1s}\frac{\partial \mathrm{R}}{\partial a_{1s}} + a_{2s}\frac{\partial \mathrm{R}}{\partial a_{2s}} + \ldots .$$

The digression on 'les fonctions de M. Jacobi' being exhausted, Cayley returns to skew symmetric determinants with the requisite

It is instructive, in connection with the matter in hand, to note that this function is expressible in terms of four Pfaffians, viz., we have

$$\Sigma \pm 12{\cdot}34 = 2\left\{ \begin{vmatrix} 12 & 13 & 14 \\ & 23 & 24 \\ & & 34 \end{vmatrix} - \begin{vmatrix} 12 & 13 & 14 \\ & 32 & 42 \\ & & 43 \end{vmatrix} + \begin{vmatrix} 21 & 31 & 41 \\ & 32 & 42 \\ & & 43 \end{vmatrix} - \begin{vmatrix} 21 & 31 & 41 \\ & 23 & 24 \\ & & 34 \end{vmatrix} \right\};$$

and thus see that, if the condition $rs = -sr$ be introduced, the result is

$$\sum_{rs=-sr} \pm 12{\cdot}34 = 8 \cdot \begin{vmatrix} 12 & 13 & 14 \\ & 23 & 24 \\ & & 34 \end{vmatrix};$$

so that the Pfaffian on the right may be defined as the eighth part of a certain Cayleyan determinant; or, in Cayley's symbols,

$$[1\,2\,3\,4] = \tfrac{1}{8}\sum_{rs=-sr} \pm 12{\cdot}34 ,$$

where the 8 is the value of $2^{\frac{1}{2}n}(1.2 \ldots . \tfrac{1}{2}n)$ when $n=4$.

Before leaving this it deserves to be noted that when Cayley came in 1889 to re-edit his writings, he appended to this paper a note in which it is stated that part of his purpose was to show "that the definition of a determinant may be so extended as to include within it the Pfaffian" (see *Collected Math. Papers*, i. p. 589).

material for proving the two theorems above referred to. The first of them, which is not new, is, in later phraseology that "*Any zero-axial skew determinant of odd order vanishes*"; and the second, which is Cayley's own, is that "*Any zero-axial skew determinant of even order is the square of a Pfaffian.*" In both cases the method of proof is that known as 'mathematical induction'; and in both cases the main auxiliary theorem used is Cauchy's regarding the expansion of a determinant according to binary products of the elements of a row and the elements of a column.

When n is odd and the elements of the first row and those of the first column are $0, \lambda_{12}, \lambda_{13}, \ldots, \lambda_{1n}$ and $0, \lambda_{21}, \lambda_{31}, \ldots, \lambda_{n1}$ respectively, he says it is easy to see that for each term having $\lambda_{1\alpha}\lambda_{\beta 1}$ for a factor, where $\alpha \neq \beta$, there exists an equal term of opposite sign having $\lambda_{1\beta}\lambda_{\alpha 1}$ for a factor; and that therefore, since $\lambda_{1\alpha}\lambda_{\beta 1} = \lambda_{1\beta}\lambda_{\alpha 1}$, these two terms must cancel each other. As for the terms which have $\lambda_{1\alpha}\lambda_{\alpha 1}$ for a factor, the co-factor is a determinant of exactly the same form as the original, but of the order $n-2$; consequently the theorem is seen to hold for any one case if it hold for the case immediately preceding. But for the case where $n=3$, the theorem is self-evident; therefore, "Tout déterminant gauche et symétrique d'un ordre *impair* est zéro."

When n is even, the determinant dealt with is purposely taken more general than one with skew symmetry, although, strange to say, Cayley calls it 'gauche et symétrique,' the elements of the first row and those of the first column being $\lambda_{\alpha\beta}, \lambda_{\alpha 2}, \lambda_{\alpha 3}, \ldots, \lambda_{\alpha n}$ and $\lambda_{\alpha\beta}, \lambda_{2\beta}, \lambda_{3\beta}, \ldots, \lambda_{n\beta}$, and his aim being to prove that such a determinant is equal to the product of two of the functions treated of in the digression, viz., $[\alpha\, 2\, 3 \ldots n]$ and $[\beta\, 2\, 3 \ldots n]$. Developing as in the preceding case, there has this time to be considered the element common to the first row and first column, viz., $\lambda_{\alpha\beta}$, the co-factor of which is seen to be a skew symmetric determinant of odd order $n-1$, and therefore, as has just been shown, is equal to zero. As for the co-factor of $-\lambda_{\alpha\alpha'}\lambda_{\beta'\beta}$, where $\lambda_{\alpha\alpha'}$ is any element of the first row except the first, and $\lambda_{\beta'\beta}$ is any element of the first column except the first, it will be found to be a determinant which Cayley again mistakenly but consistently calls 'gauche et symétrique,' obtained by giving to r all the values $2, 3, \ldots, n$ with the exception of α', and to s all the values

2,3, . . . , n, with the exception of β'. This determinant of the $(n-2)^{th}$ order is expected to be seen to be of the same kind as that with which we started, and to be temporarily admitted to be equal to

$$[\alpha'+1, \ldots, n, 2, \ldots, \alpha'-1]\cdot[\beta'+1, \ldots, n, 2, \ldots, \beta'-1].$$

The typical term of the expansion will thus be

$$\lambda_{\alpha\alpha'}[\alpha'+1, \ldots, n, 2, \ldots, \alpha'-1]\cdot\lambda_{\beta'\beta}[\beta'+1, \ldots, n, 2, \ldots, \beta'-1];$$

and the sum of all such terms

$$\begin{aligned} = &\{\lambda_{\alpha2}[34\ldots n]+\lambda_{\alpha3}[4\ldots n2]+\ldots+\lambda_{\alpha n}[23\ldots(n-1)]\} \\ \cdot &\{\lambda_{\beta2}[34\ldots n]+\lambda_{\beta3}[4\ldots n2]+\ldots+\lambda_{\beta n}[23\ldots(n-1)]\} \end{aligned}$$

and therefore

$$= [\alpha\, 2\, 3\ldots n]\cdot[\beta\, 2\, 3\ldots n].$$

This means, of course, that if the theorem holds for a determinant of order $n-2$ it will hold for the succeeding case. But in the simplest case, viz., where $n=2$, it is self-evident that the theorem holds, for the determinant then

$$\begin{aligned} &= \lambda_{\alpha\beta}\lambda_{22}-\lambda_{2\beta}\lambda_{\alpha2}, \\ &= \lambda_{\beta2}\lambda_{\alpha2}, \\ &= [\beta2]\cdot[\alpha2]; \end{aligned}$$

consequently "*Le déterminant gauche et symétrique qu'on obtient en donnant à* r *les valeurs* α,2,3, . . . , n, *et à* s *les valeurs* β,2,3, . . . , n (*où* n *est* pair) *se réduit à*

$$[\alpha\, 2\, 3\ldots n]\cdot[\beta\, 2\, 3\ldots n];$$

et en particulier, en donnant à r, s *les valeurs* 1,2, . . . , n *ce déterminant se réduit à* [1 2 3 . . . n]²".

Going back now to the expansion of the skew determinant Ω with which the paper opened, and taking for simplicity's sake * $\lambda_{rr}=1$ in every case, Cayley readily obtains,

for n even,
$$\begin{aligned} \Omega = \;&[123\ldots n]^2 \\ &+[34\ldots n]^2+[24\ldots n]^2+\ldots \\ &+[56\ldots n]^2+\ldots \\ &+\;\ldots\ldots \\ &+\;1, \end{aligned}$$

* And of course without loss of generality, as Cayley might have said.

and, for n odd,
$$\begin{aligned}\Omega = \;& [23\ldots n]^2 + [13\ldots n]^2 + \ldots \\ & + [45\ldots n]^2 + \ldots\ldots \\ & + \ldots\ldots \\ & + 1 .\end{aligned}$$

A special example of each identity is given, viz., the examples in which $n=4$ and 3 respectively. If we make a slight change in the left member, viz., write Ω in Cayley's vertical-line notation (which, by the way, considering the help it would have given, and the fact that it had been introduced six years previously, it is surprising not to find employed in this paper), these examples take the form,—

$$\begin{vmatrix} 1 & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ -\lambda_{12} & 1 & \lambda_{23} & \lambda_{24} \\ -\lambda_{13} & -\lambda_{23} & 1 & \lambda_{34} \\ -\lambda_{14} & -\lambda_{24} & -\lambda_{34} & 1 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 1 & 12 & 13 & 14 \\ -12 & 1 & 23 & 24 \\ -13 & -23 & 1 & 34 \\ -14 & -24 & -34 & 1 \end{vmatrix}$$

$$\begin{aligned} = \;& (\lambda_{12}\lambda_{34} - \lambda_{13}\lambda_{24} + \lambda_{14}\lambda_{23})^2 \\ & + \lambda^2_{12} + \lambda^2_{13} + \lambda^2_{14} + \lambda^2_{34} + \lambda^2_{24} + \lambda^2_{23} + 1 , \\ = \;& [1234]^2 + [12]^2 + [13]^2 + [14]^2 + [34]^2 + [24]^2 + [23]^2 + 1 ; \end{aligned}$$

and

$$\begin{vmatrix} 1 & \lambda_{12} & \lambda_{13} \\ -\lambda_{12} & 1 & \lambda_{23} \\ -\lambda_{13} & -\lambda_{23} & 1 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 1 & 12 & 13 \\ -12 & 1 & 23 \\ -13 & -23 & 1 \end{vmatrix}$$

$$\begin{aligned} &= \lambda^2_{23} + \lambda^2_{13} + \lambda^2_{12} + 1 . \\ &= [23]^2 + [13]^2 + [12]^2 + 1 . \end{aligned}$$

SPOTTISWOODE (1851, 1853).

[ELEMENTARY THEOREMS RELATING TO DETERMINANTS. By William Spottiswoode, M.A., of Balliol College, Oxford, viii + 63 pp. London, 1851. Second edition, as an article in *Crelle's Journ.*, li. pp. 209–271, 328–381.]

In this the earliest of modern text-books on Determinants, a special section (§ ix. pp. 46–51 ; or § vi. pp. 260–266 in second edition) is set apart with the heading "On Skew Determinants." As a matter of fact, however, it is only the latter half of the section which at present concerns us, as the other half deals in

reality with Cayley's determinant solution of the problem of orthogonal transformation.

In a sense the mode of treatment is indirect, the general skew determinant being viewed, not as a separate entity, but in its relation to a set of linear equations, the coefficients of which are its elements. The set of equations is

$$\left.\begin{array}{l}(11)x_1 + (12)x_2 + \ldots + (1n)x_n = u_1 \\ (21)x_1 + (22)x_2 + \ldots + (2n)x_n = u_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (n1)x_1 + (n2)x_2 + \ldots + (nn)x_n = u_n\end{array}\right\} ,$$

where it has to be remembered that in every instance $(rr) = 0$ and $(rs) + (sr) = 0$. The right-hand members of what he calls the "derived" set are $v_1, v_2, \ldots, v_n$; that is to say, there exists simultaneously with the original the set

$$\left.\begin{array}{l}(11)x_1 + (21)x_2 + \ldots + (n1)x_n = v_1 \\ (12)x_1 + (22)x_2 + \ldots + (n2)x_n = v_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (1n)x_1 + (2n)x_2 + \ldots + (nn)x_n = v_n\end{array}\right\}$$

whose determinant is got from the determinant of the former set by the change of rows into columns, and may therefore be denominated by the same symbol Δ. Solving the two sets of equations we have

$$\left.\begin{array}{l}x_1\Delta = [11]u_1 + [12]u_2 + \ldots + [1n]u_n, \\ x_2\Delta = [21]u_1 + [22]u_2 + \ldots + [2n]u_n, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n\Delta = [n1]u_1 + [n2]u_2 + \ldots + [nn]u_n,\end{array}\right\}$$

and

$$\left.\begin{array}{l}x_1\Delta = [11]v_1 + [21]v_2 + \ldots + [n1]v_n, \\ x_2\Delta = [12]v_1 + [22]v_2 + \ldots + [n2]v_n, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n\Delta = [1n]v_1 + [2n]v_2 + \ldots + [nn]v_n\end{array}\right\} * ,$$

where, be it remarked, it would have been much better if in every case the coefficients of u_r and v_r had been interchanged, for then

* There is herein used the fact, first noted by Rothe in 1800, that the cofactor of rs in any determinant is equal to the cofactor of sr in the conjugate determinant.

$[rs]$ would have stood for the cofactor of (rs) in Δ. From these by addition and subtraction and by utilizing the fact that $u_r + v_r = 0$ † two others are obtained, viz.,

$$\left.\begin{array}{llll} 2x_1\Delta = & 0 & +([12]-[21])u_2 + \ldots & +([1n]-[n1])u_n \\ 2x_2\Delta = & ([21]-[12])u_1 + & 0 \quad + \ldots & +([2n]-[n2])u_n \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ 2x_n\Delta = & ([n1]-[1n])u_1 & +([n2]-[2n])u_2 + \ldots + & 0 \end{array}\right\}$$

and

$$\left.\begin{array}{llll} 0 = & 2[11]u_1 & +([12]+[21])u_2 + \ldots & +([1n]+[n1])u_n \\ 0 = & ([21]+[12])u_1 + & 2[22]u_2 + \ldots & +([2n]+[n2])u_n \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ 0 = & ([n1]+[1n])u_1 & +([n2]+[2n])u_2 + \ldots + & 2[nn]u_n \end{array}\right\}.$$

Then follows the very curious sentence—curious, that is to say, logically—

"The comparison of these three systems gives either

$$\Delta = 0$$

$$\left.\begin{array}{cccc} * & [12]=[21] & \ldots & [1n]=[n1] \\ [21]=[12] & * & \ldots & [2n]=[n2] \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ [n1]=[1n] & [n2]=[2n] & \ldots & * \end{array}\right\}$$

or

$$\left.\begin{array}{cccc} [11=0] & [12]+[21]=0 & \ldots & [1n]+[n1]=0 \\ [21]+[12]=0 & [22]=0 & \ldots & [2n]+[n2]=0 \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ [n1]+[1n]=0 & [n2]+[2n]=0 & \ldots & [nn]=0 \end{array}\right\};$$

and consequently either a symmetrical skew determinant of an even order or a" [symmetrical skew] "determinant of an odd order vanishes."

Temporarily setting aside the latter portion of this sentence we see that what is considered to be proved is the proposition that *If* Δ *be a zero-axial skew determinant, then either*

$$(1) \quad \Delta = 0 \textit{ and } [rs]=[sr],$$
$$\textit{or } (2) \quad [rr]=0 \textit{ and } [rs]=-[sr].$$

† Along with this fact Spottiswoode associates the statements that $u_1+u_2+\ldots+u_n=0$, $v_1+v_2+\ldots+v_n=0$, which are manifestly incorrect.

How the said latter portion—that is to say, the deduction from this—can be justified is a mystery; but of course if it be granted there is no objection to the cogency of the next step in the reasoning, which is worded as follows:—

> "But since it is found on trial that for $n=1, 3, \ldots, \Delta$ vanishes, while for $n=2, 4, \ldots$, it does not, the following theorems may be enunciated:—
>
> "Theorem XIV. *A symmetrical skew determinant of an odd order in general vanishes, and the system has for its inverse an unsymmetrical skew system.*
>
> "Theorem XV. *A symmetrical skew determinant of an even order does not in general vanish, but the system has for its inverse a symmetrical skew system.*"

The name, however, given to the "inverse system" in the first case when, as we have seen, $[rs]=[sr]$ is clearly inappropriate; and it is not improved in the second edition by alteration into "quadratic skew," the fact being that the system is not skew at all, but is symmetric with respect to the principal diagonal, or, in later phraseology, is *axisymmetric*.

The treatment of the next theorem taken up is happier than the foregoing, and is after the outset no less fresh. Taking an even-ordered skew determinant with zeros in the principal diagonal he develops it according to products of an element of the first row and an element of the first column, the result being written in the form

$$\begin{vmatrix} * & 12 & \ldots & 1n \\ 21 & * & & 2n \\ \cdot & \cdot & \cdot & \cdot \\ n1 & n2 & & * \end{vmatrix} = (12)^2 \begin{vmatrix} * & 34 & \ldots & 3n \\ 43 & * & \ldots & 4n \\ \cdot & \cdot & \cdot & \cdot \\ n3 & n4 & \ldots & * \end{vmatrix} + 2\,(12)\,(13) \begin{vmatrix} 34 & 35 & \ldots & 32 \\ * & 45 & \ldots & 42 \\ \cdot & \cdot & \cdot & \cdot \\ n4 & n5 & \ldots & n2 \end{vmatrix} + \ldots$$

where, be it observed, the second typical term on the right has been altered from

$$-2\,(12)\,(13) \begin{vmatrix} 32 & 34 & \ldots & 3n \\ 42 & * & \ldots & 4n \\ \cdot & \cdot & \cdot & \cdot \\ n2 & n4 & \ldots & * \end{vmatrix}$$

by the translation of the first column to the last place. The determinant in this typical term is then further transformed into the square root of the product of two determinants like that in the term preceding it, the steps of the reasoning being—

$$\begin{vmatrix} 32 & 34 & \dots & 3n \\ 42 & * & \dots & 4n \\ \cdot & \cdot & \cdot & \cdot \\ n2 & n4 & \dots & * \end{vmatrix}^2 = \begin{vmatrix} 23 & 24 & \dots & 2n \\ 43 & * & & 4n \\ \cdot & \cdot & \cdot & \cdot \\ n3 & n4 & & * \end{vmatrix} \cdot \begin{vmatrix} 32 & 34 & \dots & 3n \\ 42 & * & \dots & 4n \\ \cdot & \cdot & \cdot & \cdot \\ n2 & n4 & \dots & * \end{vmatrix},$$

$$= \begin{vmatrix} * & 24 & \dots & 2n \\ 43 & * & \dots & 4n \\ \cdot & \cdot & \cdot & \cdot \\ n3 & n4 & \dots & * \end{vmatrix} \cdot \begin{vmatrix} * & 34 & \dots & 3n \\ 42 & * & \dots & 4n \\ \cdot & \cdot & \cdot & \cdot \\ n2 & n4 & \dots & * \end{vmatrix},$$

the deletion of 23 and 32 in the last step being warranted by the fact that their cofactors are determinants similar to the original but of odd order $n-3$, and therefore have the value zero. The development as thus changed has the form of the square of a polynomial; and consequently by extracting the square root there results

$$\begin{vmatrix} * & 12 & \dots & 1n \\ 21 & * & \dots & 2n \\ \cdot & \cdot & \cdot & \cdot \\ n1 & n2 & \dots & * \end{vmatrix}^{\frac{1}{2}} = 12 \cdot \begin{vmatrix} * & 34 & \dots & 3n \\ 43 & * & \dots & 4n \\ \cdot & \cdot & \cdot & \cdot \\ n3 & n4 & \dots & * \end{vmatrix}^{\frac{1}{2}} + 13 \cdot \begin{vmatrix} * & 45 & \dots & 42 \\ 54 & * & \dots & 52 \\ \cdot & \cdot & \cdot & \cdot \\ 24 & 25 & \dots & * \end{vmatrix}^{\frac{1}{2}} + \dots$$

This, according to the point of view, will be recognised either as Cayley's theorem that an even-ordered skew determinant with zeros in the principal diagonal is a *square*, or as the theorem in Pfaffians formulated by Cayley and which in Jacobi's notation would be written

$$[123 \dots n] = 12\,[34 \dots n] + 13\,[45 \dots n2] + 14\,[56 \dots n23] + \dots$$

The rest of the section or chapter deals with Cayley's extension of this to skew determinants whose principal elements are not zeros, the notation employed being the same.

CAYLEY (1851).

[" On the Theory of Permutants." *Camb. and Dub. Math. Journ.* vii. pp. 40–51; or *Collected Math. Papers,* ii. pp. 16–26.]

By this time the widened definition of a determinant which Cayley had given in his paper of 1847 had been exploited to a certain extent, and had been found profitable both by himself and his fellow-worker Sylvester. The paper we have now come to, however, is the only one of the series that for the present concerns us.* In it he implicitly discards his former usage of the word "determinant" in any wider sense than that employed by his predecessors; adopts instead the word "*permutant*" as suggested by Sylvester, and in working out the theory of the general functions under this name assigns to determinants and Pfaffians their proper niches in the new structure, the scheme of classification being

Permutants { (A) (no name) ; (B) Intermutants (or hyperdeterminants) }

{ (a) Pfaffians ; (b) Commutants { (β) Determinants. }

CAYLEY (1854).

[" Recherches ultérieures sur les déterminants gauches." *Crelle's Journ.*, l. pp. 299–313; or *Collected Math. Papers*, ii. pp. 202–205.]

The development with which his paper of 1847 closes is here recalled and repeated for the case where the skew determinant is of the 5th order and the elements of the diagonal are specialized, the form in which the identity appears being

* All of them fall to be dealt with when giving the history of the development of the theory of determinants in general.

$$
\begin{aligned}
\overline{12345 \mid 13345} = \quad & 11 \cdot 22 \cdot 33 \cdot 44 \cdot 55 \\
& + 11 \cdot 22 \cdot 33 \cdot (45)^2 \\
& + 11 \cdot 22 \cdot 44 \cdot (35)^2 \\
& + 11 \cdot 22 \cdot 55 \cdot (34)^2 \\
& + 11 \cdot 33 \cdot 44 \cdot (25)^2 \\
& + 11 \cdot 33 \cdot 55 \cdot (24)^2 \\
& + 11 \cdot 44 \cdot 55 \cdot (23)^2 \\
& + 22 \cdot 33 \cdot 44 \cdot (15)^2 \\
& + 22 \cdot 33 \cdot 55 \cdot (14)^2 \\
& + 22 \cdot 44 \cdot 55 \cdot (13)^2 \\
& + 33 \cdot 44 \cdot 55 \cdot (12)^2 \\
& + 11 \cdot (2345)^2 \\
& + 22 \cdot (1345)^2 \\
& + 33 \cdot (1245)^2 \\
& + 44 \cdot (1235)^2 \\
& + 55 \cdot (1234)^2,
\end{aligned}
$$

where the symbol on the left stands for the determinant whose elements are 11, 12, . . . , 21, 22, . . . and the peculiarity of skewness is understood but not expressed. Had the specialization of the elements of the diagonal been as before, the development would clearly have been

$$
\begin{aligned}
& 1 \\
& + (45)^2 + (35)^2 + (34)^2 + (25)^2 + (24)^2 + (23)^2 + (15)^2 + (14)^2 + (13)^2 + (12)^2 \\
& \quad + (2345)^2 + (1345)^2 + (1245)^2 + (1235)^2 + (1234)^2,
\end{aligned}
$$

which, if the order be reversed, agrees exactly with the result of putting $n = 5$ in the identity towards the end of the paper of 1846. By way of explanation Cayley adds the sentence "Les expressions 12, 1234, etc, à droite sont ici des *Pfaffiens*,"—which is noteworthy as being the first intimation that he desired "les fonctions de M. Jacobi," as he had formerly called them, to be known by the name of the mathematician whose integration-method had led Jacobi to the discovery of them. The change is easily accounted for by the fact that it was more appropriate to attach Jacobi's name to another class of determinants which were of greater importance and to which Jacobi had given far more attention.

Immediately following this there comes the announcement :—

"J'ai trouvé récemment une formule analogue pour le développement d'un *déterminant gauche* bordé, tel que

$$\overline{\alpha 1234 \mid \beta 1234} = \begin{vmatrix} \alpha\beta & \alpha 1 & \alpha 2 & \alpha 3 & \alpha 4 \\ 1\beta & 11 & 12 & 13 & 14 \\ 2\beta & 21 & 22 & 23 & 24 \\ 3\beta & 31 & 32 & 33 & 34 \\ 4\beta & 41 & 42 & 43 & 44 \end{vmatrix};$$

Cette formule est :—

$$\begin{aligned} \overline{\alpha 1234 \mid \beta 1234} = \;& \alpha\beta \cdot 11 \cdot 22 \cdot 33 \cdot 44 \\ & \left.\begin{aligned} &+\alpha\beta \cdot 12 \cdot 12 \cdot 33 \cdot 44 \\ &+\alpha\beta \cdot 13 \cdot 13 \cdot 22 \cdot 44 \\ &+\alpha\beta \cdot 14 \cdot 14 \cdot 22 \cdot 33 \\ &+\alpha\beta \cdot 23 \cdot 23 \cdot 11 \cdot 44 \\ &+\alpha\beta \cdot 24 \cdot 24 \cdot 11 \cdot 33 \\ &+\alpha\beta \cdot 34 \cdot 34 \cdot 11 \cdot 22 \end{aligned}\right\} \\ & +\alpha\beta 1234 \cdot 1234^* \\ & \left.\begin{aligned} &+\alpha 1 \cdot \beta 1 \cdot 22 \cdot 33 \cdot 44 \\ &+\alpha 2 \cdot \beta 2 \cdot 11 \cdot 33 \cdot 44 \\ &+\alpha 3 \cdot \beta 3 \cdot 11 \cdot 22 \cdot 44 \\ &+\alpha 4 \cdot \beta 4 \cdot 11 \cdot 22 \cdot 33 \end{aligned}\right\} \\ & \left.\begin{aligned} &+\alpha 123 \cdot \beta 123 \cdot 44 \\ &+\alpha 124 \cdot \beta 124 \cdot 33 \\ &+\alpha 134 \cdot \beta 134 \cdot 22 \\ &+\alpha 234 \cdot \beta 234 \cdot 11 \end{aligned}\right\}. \end{aligned}$$

Naturally enough it is noted by Cayley that the writing of $\alpha = \beta = 5$ gives us the less general theorem with which we started; but he does not explain why a third way of arranging the terms of the development is adopted. Stranger still, he does not remark on the fact that by making 11, 22, 33, 44 all vanish there is obtained the identity

$$\underset{rs=-sr,\ rr=0}{\overline{\alpha 1234 \mid \beta 1234}} = \alpha\beta 1234 \cdot 1234,$$

which is the twin theorem to one given in his previous paper

* A serious misprint in the original is here corrected.

regarding a bordered skew symmetrical determinant of *even* order. It will be remembered, however, that in the statement of this latter theorem, the peculiar narrow use of the word 'bordé' did not occur.

Although what may be called Part Second of the paper (pp. 301, 302) may seem at first sight to concern something else, it really only draws attention to the fact that *the minors* (by which he means those afterwards named *primary* minors) *of a skew determinant are themselves skew, being "gauches ordinaires" when their cofactor in the original determinant is of the form* rr, *and "gauches bordés" when their cofactor is of the form* rs. Considerable space is occupied in verifying by two examples that the same result will be reached whether we apply the theorem of Part First directly to

$$\overline{123\ldots n \mid 123\ldots n}$$

or to the primary minors in its equivalent

$$11\cdot\overline{23\ldots n \mid 23\ldots n} - 12\cdot\overline{23\ldots n \mid 13\ldots n} + \ldots\ldots$$

What may be called Part Third (pp. 303–305) is very forbidding, by reason of the defective mode of exposition and of the awkwardness of the notation employed. Probably this accounts for the fact that the interesting theorem which it contains has never emerged until now from its place of sepulture. A portion of it must of necessity be given verbatim, if only for the purpose of preserving historical colour. It commences—

> "Je remarque que le nombre des termes du développement (p. 299) du déterminant gauche est toujours une *puissance de* 2, et que de plus, ce nombre se réduit à la moitié, en réduisant à zéro un terme quelconque *aa*. Mais outre cela, le déterminant prend dans cette supposition la forme de déterminant [gauche] d'un ordre inférieur de l'unité. Je considère par example le déterminant gauche $\overline{123 \mid 123}$. En y faisant $33=0$ et en accentuant, pour y mettre plus de clarté, tous les symboles, on trouve

$$\overline{123 \mid 123'} = 11'\cdot(23')^2 + 22'\cdot(13')^2.$$

De là, en écrivant

$$11 = 13'\cdot 11', \qquad 12 = 11'\cdot 23',$$
$$22 = 13'\cdot 22',$$

on obtient

$$\overline{12 \mid 12} = 11\cdot 22 + (12)^2,$$
$$= 11'\cdot\{22'\cdot(13')^2 + 11'\cdot(23')^2\},$$

c'est à dire

$$\overline{12 \mid 12} = 11'\cdot\overline{123 \mid 123'}\cdot$$

On a de même

$$\overline{1234 \mid 1234'} = 11'\cdot 22'\cdot(34')^2 + 11'\cdot 33'\cdot(24')^2 + 22'\cdot 33'\cdot(14')^2 + (1234')^2,$$

et delà, en écrivant

$$\begin{array}{lll} 11 = 14'\cdot 11', & 12 = 11'\cdot 24', & 23 = 1234', \\ 22 = 14'\cdot 22', & 13 = 11'\cdot 34', & \\ 33 = 14'\cdot 33', & & \end{array}$$

on obtient

$$\overline{123 \mid 123} = 11\cdot 22\cdot 33 + 11\cdot(23)^2 + 22\cdot(31)^2 + 33\cdot(12)^2,$$
$$= 11'\cdot 14' \left\{ \begin{array}{l} 22'\cdot 33'\cdot(14')^2 + (1234')^2 \\ \qquad\qquad + 11'22'\cdot(34')^2 + 11'\,33'(24')^2 \end{array} \right\}$$

c'est à dire

$$\overline{123 \mid 123} = 11'\cdot 14'\cdot\overline{1234 \mid 1234'}."$$

The remainder is devoted to the next two cases, the verification of which, of course, occupies still more space. The theorem thus dealt with may be roughly described as giving *the transformation of a skew determinant, having one zero element in its main diagonal, into a skew determinant of the next lower order;* and in a notation which needs no explanation and which was perfectly familiar to Cayley at the time, the four examples may be written thus:—

$$\begin{vmatrix} 11 & 12 & 13 \\ -12 & 22 & 23 \\ -13 & -23 & \cdot \end{vmatrix} = \begin{vmatrix} 11\cdot13 & 11\cdot23 \\ -11\cdot23 & 22\cdot13 \end{vmatrix} \div 11,$$

$$\begin{vmatrix} 11 & 12 & 13 & 14 \\ -12 & 22 & 23 & 24 \\ -13 & -23 & 33 & 34 \\ -14 & -24 & -34 & \cdot \end{vmatrix} = \begin{vmatrix} 11\cdot14 & 11\cdot24 & 11\cdot34 \\ -11\cdot24 & 22\cdot14 & [1234] \\ -11\cdot34 & -[1234] & 33\cdot14 \end{vmatrix} \div 11\cdot14,$$

$$\begin{vmatrix} 11 & 12 & 13 & 14 & 15 \\ -12 & 22 & 23 & 24 & 25 \\ -13 & -23 & 33 & 34 & 35 \\ -14 & -24 & -34 & 44 & 45 \\ -15 & -25 & -35 & -45 & \cdot \end{vmatrix} = \begin{vmatrix} 11\cdot15 & 11\cdot25 & 11\cdot35 & 11\cdot45 \\ -11\cdot25 & 22\cdot15 & [1235] & [1245] \\ -11\cdot35 & -[1235] & 33\cdot15 & [1345] \\ -11\cdot45 & -[1245] & -[1345] & 44\cdot15 \end{vmatrix} \div 11\cdot(15)^2,$$

$$\begin{vmatrix} 11 & 12 & \ldots & 15 & 16 \\ -12 & 22 & \ldots & 25 & 26 \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot \\ -15 & -25 & \ldots & 55 & 56 \\ -16 & -26 & \ldots & -56 & \cdot \end{vmatrix} = \begin{vmatrix} 11\cdot16 & 11\cdot26 & 11\cdot36 & 11\cdot46 & 11\cdot56 \\ -11\cdot26 & 22\cdot16 & [1236] & [1246] & [1256] \\ -11\cdot36 & -[1236] & 33\cdot16 & [1346] & [1356] \\ -11\cdot46 & -[1246] & -[1346] & 44\cdot16 & [1456] \\ -11\cdot56 & -[1256] & -[1356] & -[1456] & 55\cdot16 \end{vmatrix} \div 11\cdot(16)^3.$$

Of course, this mode of writing does not at once suggest any better mode of proof, but it makes clear the general theorem, which consequently may be enunciated as follows :—

"*A skew determinant of the* nth *order which has a zero for the last element of its main diagonal may, if multiplied by* $11\cdot(\mathrm{n})^{n-3}$ *be transformed into a skew determinant of the* $(\mathrm{n}-1)^{th}$ *order, which has for its first row the last column of the original determinant multiplied by* 11, *for its main diagonal the main diagonal of the original determinant multiplied by* 1n, *and for the element in every other place* rs *situated between these two lines the Pfaffian* [1rsn].

The rest of the paper deals with *inverse matrices*, and with the application of them to the problem afterwards known as the *automorphic transformation of a quadric.*

BRIOSCHI (1854).

[La Teorica dei Determinanti, e le sue principali Applicazioni. Del Dr. Francisco Brioschi. viii + 116 pp. Pavia, 1854.
Translation into French, by Combescure, ix + 216 pp. Paris, 1856.
Translation into German, by Schellbach, vii + 102 pp. Berlin, 1856.]

In this, the second text-book, the same importance is given to skew determinants as in Spottiswoode, the first part of the eighth section (pp. 55–72) being devoted to them under the heading "Dei determinanti *gobbi*," which Schellbach translates by *überschlagen*. The arrangement and treatment of the matter, however, are much more logical, zero-axial skew determinants being taken first, then the functions connected with these, viz., Pfaffians, then skew determinants which are not zero-axial, and lastly the use of skew determinants in the consideration of the problem of orthogonal transformation.

The precedence given to determinants which are "gobbi simmetrici" over those which are "puramente gobbi" is explained at the outset by reference to Cayley's theorem regarding the expressibility of the latter in terms of the former, the quite general theorem from which Cayley's immediately follows being carefully enunciated thus:—

> "Indicando con P_o il determinante nel quale si pongano equali a zero gli elementi principali; e con $({}^m P_{ii})_o$ un determinante minore principale delle' m-esimo ordine del determinante P nel quale siensi annullati gli elementi principali si ha:—
>
> $$\left.\begin{aligned} P = P_o + \sum\nolimits_r a_{rr}({}^1P_{ii})_o + \sum\nolimits_r \sum\nolimits_s a_{rr}a_{ss}({}^2P_{ii})_o \\ + \ldots + a_{11}a_{22}\ldots a_{nn} \end{aligned}\right\} .$$

The proof given of Jacobi's theorem regarding the value of an odd-ordered skew determinant with zero, in the principal diagonal is essentially the same as Cayley's proof, but fuller and clearer. The proof of the corresponding theorem for a determinant of even order resembles Spottiswoode's, the difference lying mainly in the

use of the notation of differential-quotients in specifying the minors of the determinant. Denoting the determinant of even order by P, he starts with the development—

$$P = -a^2_{1r}\frac{\partial^2 P}{\partial a_{1r}\,\partial a_{r1}} - a^2_{1s}\frac{\partial^2 P}{\partial a_{1s}\,\partial a_{s1}} \pm 2a_{1r}a_{1s}\frac{\partial^2 P}{\partial a_{1r}\,\partial a_{1s}}$$
$$- \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; .$$

Then as a previously obtained general identity, originally due to Jacobi, viz.,

$$P\frac{\partial^2 P}{\partial a_{rs}\,\partial a_{pq}} = \frac{\partial P}{\partial a_{rs}}\cdot\frac{\partial P}{\partial a_{pq}} - \frac{\partial P}{\partial a_{ps}}\cdot\frac{\partial P}{\partial a_{rq}},$$

gives in this special case the identities

$$P\frac{\partial^2 P}{\partial a_{1r}\,\partial a_{r1}} = \frac{\partial P}{\partial a_{1r}}\cdot\frac{\partial P}{\partial a_{r1}}, \quad P\frac{\partial^2 P}{\partial a_{1s}\,\partial a_{s1}} = \frac{\partial P}{\partial a_{1s}}\cdot\frac{\partial P}{\partial a_{s1}},$$

$$P\frac{\partial^2 P}{\partial a_{1r}\,\partial a_{s1}} = \frac{\partial P}{\partial a_{1r}}\cdot\frac{\partial P}{\partial a_{s1}},$$

because the cofactor, awkwardly denoted by $\partial P/\partial a_{ii}$, of any vanishing element a_{ii} in the principal diagonal is zero in accordance with the preceding theorem of Cayley's. From the first two of these we have

$$P^2 \cdot \frac{\partial^2 P}{\partial a_{1r}\,\partial a_{r1}} \cdot \frac{\partial^2 P}{\partial a_{1s}\,\partial a_{s1}} = \frac{\partial P}{\partial a_{1r}}\cdot\frac{\partial P}{\partial a_{r1}} \times \frac{\partial P}{\partial a_{1s}}\cdot\frac{\partial P}{\partial a_{s1}};$$

the right side of which can be changed into

$$\left(\frac{\partial P}{\partial a_{1r}}\cdot\frac{\partial P}{\partial a_{s1}}\right)^2$$

by reason of the fact that for a determinant such as P we have in every case

$$\frac{\partial P}{\partial a_r} = -\frac{\partial P}{\partial a_{sr}}.$$

But from the third identity above, by squaring, we obtain on the right the same expression; so that there thus results

$$\left(\frac{\partial^2 P}{\partial a_{1r}\,\partial a_{s1}}\right)^2 = \frac{\partial^2 P}{\partial a_{1r}\,\partial a_{r1}}\cdot\frac{\partial^2 P}{\partial a_{1s}\,\partial a_{s1}},$$

—an equation which exactly expresses the property that the

determinant P is a square ("nella quale equazione trovasi appunto espressa la proprietà che il determinante P è un quadrato").

On looking now to the development with which the demonstration opened Brioschi is led to an expression for the square in question, viz.:

$$P = \left\{ \pm a_{12}\left(\frac{\partial^2 P}{\partial a_{11}\,\partial a_{22}}\right)^{\frac{1}{2}} \pm a_{13}\left(\frac{\partial^2 P}{\partial a_{11}\,\partial a_{33}}\right)^{\frac{1}{2}} \pm \cdots a_{1n}\left(\frac{\partial^2 P}{\partial a_{11}\,\partial a_{nn}}\right)^{\frac{1}{2}} \right\}^2,$$

or, more generally,

$$P = \left\{ \sum\nolimits_{1s} \pm a_{rs}\left(\frac{\partial^2 P}{\partial a_{rr}\,\partial a_{ss}}\right)^{\frac{1}{2}} \right\}^2,$$

where he notes that in every case $a_{rr} = 0$ and $\partial^2 P/\partial a_{rr}\,\partial a_{ss}$, being a determinant of the same kind as P, is a square. The example added is

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{vmatrix} = \left\{ a_{12}\begin{vmatrix} 0 & a_{34} \\ a_{43} & 0 \end{vmatrix}^{\frac{1}{2}} - a_{13}\begin{vmatrix} 0 & a_{24} \\ a_{42} & 0 \end{vmatrix}^{\frac{1}{2}} + a_{14}\begin{vmatrix} 0 & a_{23} \\ a_{32} & 0 \end{vmatrix}^{\frac{1}{2}} \right\}^2,$$

$$= (a_{12}a_{34} - a_{13}a_{24} + a_{14}\,a_{23})^2,$$

where the difficulty of the ambiguous sign, although presenting itself more prominently than in the general demonstration, is not referred to.

The new function H, which is the square root of P, is next studied. Differentiating both sides of the equation of relationship Brioschi obtains

$$\frac{\partial P}{\partial a_{rs}} = H\frac{\partial H}{\partial a_{rs}},\,*$$

where the inconvenience of the differential notation comes out more strikingly than before, the differential-quotient on the left being used conventionally to denote a certain minor of P, and the differentiation on the right being real. By squaring we have

$$\left(\frac{\partial P}{\partial a_{rs}}\right)^2 = P\left(\frac{\partial H}{\partial a_{rs}}\right)^2,$$

* Since the left member is what Cayley called a "bordered skew symmetric determinant"; and since, as Jacobi noted, a differential-quotient of H with respect to one of its elements is a function of the same kind as H, we have here one half of Cayley's proposition that *a bordered skew symmetric determinant is expressible as the product of two Pfaffians.*

and since, as we have seen, it is permissible to substitute

$$\mathrm{P}\frac{\partial^2\mathrm{P}}{\partial a_{rr}\partial a_{ss}} \text{ for } \left(\frac{\partial \mathrm{P}}{\partial a_{rs}}\right)^2$$

there results

$$\left(\frac{\partial^2\mathrm{P}}{\partial a_{rr}\partial a}\right)^{\frac{1}{2}} = \pm\frac{\partial \mathrm{H}}{\partial a_{rs}}\,;$$

so that the expansion for P above obtained may be altered into

$$\mathrm{P} = \left\{\sum\nolimits_s\left(a_{rs}\frac{\partial \mathrm{H}}{\partial a_{rs}}\right)\right\}^2,$$

from which by extraction of the square root we have

$$\mathrm{H} = \sum\nolimits_s\left(a_{rs}\frac{\partial \mathrm{H}}{\partial a_{rs}}\right).$$

This will be recognised as a third mode of writing an already well-known result, and, as Brioschi notes, gives a property of the function H similar to a property of determinants ("la quale equazione contiene una proprietà della funzione H analoga ad una nota dei determinanti").

From this he passes to what he calls the characteristic property of H, viz., its change of sign consequent upon the transposition of two indices. Calling H′ what H becomes when r and s are interchanged, he notes that in those terms of H in which the element a_{rs} occurs there can be no other element with the same indices, and that therefore

$$\frac{\partial \mathrm{H}}{\partial a_{rs}} = -\frac{\partial \mathrm{H}'}{\partial a_{rs}}.$$

Then since the same interchange made in P leaves P in reality unaltered,—that is to say, since $\mathrm{H}^2 = \mathrm{H}'^2$,—he obtains

$$\mathrm{H}\frac{\partial \mathrm{H}}{\partial a_{rs}} = \mathrm{H}'\frac{\partial \mathrm{H}'}{\partial a_{rs}}\,;$$

and, it having been shown that the two differential-quotients here appearing are of opposite signs, it follows that so also are H and H′.

Lastly, he passes on to skew determinants in general; and, using

the theorem and notation introduced at the outset, he writes Cayley's propositions in the form—

$$n \text{ even,} \qquad \mathrm{P} = \mathrm{P}_o + \sum_r \sum a_{rr}a_{ss}({}^2p_{ii})_o + \ldots + a_{11}a_{22} \ldots a_{nn},$$

$$n \text{ odd,} \qquad \mathrm{P} = \sum a_{rr}({}^1p_{ii})^o + \ldots + a_{11}a_{22} \ldots a_{nn},$$

which, he says, when the principal elements are all unity become

$$n \text{ even,} \qquad \mathrm{P} = \mathrm{P}_o + \sum_i ({}^2p_{ii})_o + \ldots + 1,$$

$$n \text{ odd,} \qquad \mathrm{P} = \sum_i ({}^1p_{ii})_o + \sum_i ({}^3p_{ii})_o + \ldots + 1,$$

the development now being in each case a sum of squares, as all the minors appearing in it are even-ordered.

CAYLEY (1857).

[Théorème sur les déterminants gauches. *Crelle's Journ.*, lv. pp 277, 278; or *Collected Math. Papers*, iv. pp. 72, 73.]

This is practically a note to rectify the oversight made in the paper of 1854, where, as has been pointed out, he omitted to draw attention to the case in which the skew determinant submitted to the operation of 'bordering' has zeros for the elements of the principal diagonal.

> "Un déterminant," he now says, "de cette espèce se réduit toujours au produit de deux *Pfaffiens*. En effet en écrivant dans les exemples $11 = 22 = 33 = 44 = 0$, on obtient :
>
> $$\overline{\alpha 123 \,|\, \beta 123} = \alpha 123 \cdot \beta 123,$$
>
> $$\overline{\alpha 1234 \,|\, \beta 1234} = \alpha\beta 1234 \cdot 1234,$$
>
> et de même pour un déterminant gauche et symétrique bordé quelconque, suivant que l'ordre du déterminant est pair ou impair."

To this there is added the suggestive commentary :—

> "Je remarque à propos de cela, que dans le cas d'un déterminant d'ordre pair, le terme $\alpha\beta$ est multiplié par un mineur premier lequel (comme déterminant gauche et symétrique d'ordre impair) se réduit à zero ; le déterminant

ne contient donc pas ce term $\alpha\beta$, et sera par conséquent fonction lineo-linéaire des quantités $\alpha 1$, $\alpha 2$, etc., et 1β, 2β, etc. ; de manière qu'on ne saurait être surpris de voir ce déterminant se présenter sous là forme d'un produit de deux facteurs, dont l'un est fonctioı. linéaire de $\alpha 1$, $\alpha 2$, etc., et l'autre fonction linéaire de 1β, 2β, etc. Mais pour un déterminant d'ordre impair, le coefficient du terme $\alpha\beta$ ne se réduit pas à zéro; en supposant donc que le déterminant puisse s' exprimer comme produit de deux facteurs, il est nécessaire que l'un de ces facteurs soit (comme le déterminant même) fonction linéaire de $\alpha\beta$ et lineo-linéaire de $\alpha 1$, $\alpha 2$, etc., et 1β, 2β, etc. : de cette manière on se rend compte de la différence de la forme des facteurs, qui a lieu dans les deux cas dont il s'agit."

It is finally pointed out that by writing $\beta = \alpha$ we are brought back to

$$\overline{\alpha 123 \mid \alpha 123} = (\alpha 123)^2,$$
$$\overline{\alpha 1234 \mid \alpha 1234} = 0:$$

—"la propriété fondamentale des déterminants gauches et symétriques." There is again, however, an oversight here, for the element $\alpha\alpha$ is taken to be equal to 0, whereas it is only necessarily so in the second case.

BALTZER (1857).

[THEORIE UND ANWENDUNG DER DETERMINANTEN. Mit Beziehung auf die Originalquellen. Dargestellt von Dr. Richard Baltzer. vi + 129 pp. Leipzig, 1857.]

Following his two predecessors Baltzer also assigned a separate section of his text-book to skew determinants, but without giving them any special designation of his own or even taking over that used by Schellbach. The title of the section (§ 8, pp. 29–34) is thus a little lengthy, viz., "*Determinante eines Systems von Elementen, unter denen die correspondirenden* a_{ik} *und* a_{ki} *entgegengesetzt gleich sind.*"

It must be noted, however, that before this section is reached some theorems which strictly belong to the subject of the section

have been already dealt with. These are in the first place (§ 3, 8; p. 12) Jacobi's theorem regarding the vanishing of a zero-axial skew determinant of odd order, and Spottiswoode's theorems regarding conjugate elements of the adjugate or inverse of a zero-axial skew determinant, the mode of proof for all being that used by Jacobi for his own theorem, viz., the multiplication of all rows or all columns by -1, and then comparing the resulting determinant with the original. In the second place (§ 3, 10; p. 13) we have Brioschi's theorem regarding the differential-quotient of a zero-axial skew determinant of even order, and a suggestive proof of the same which it is desirable to note. It is as follows:—Let the determinant

$$\begin{vmatrix} a_{11} & \cdots\cdots & a_{1n} \\ \cdots & \cdots\cdots & \cdots \\ a_{n1} & \cdots\cdots & a_{nn} \end{vmatrix}.$$

be denoted by Δ, and the cofactor of a_{rs} in Δ by A_{rs}. Then, bearing in mind that Δ is a function of a_{rs} and that a_{sr} is not independent of a_{rs}, we have

$$\begin{aligned}\frac{\partial\Delta}{\partial a_{rs}} &= A_{rs} + A_{sr}\frac{\partial a_{sr}}{\partial a_{rs}}, \\ &= A_{rs} - A_{sr}, \text{ because } a_{sr} = -a_{rs}.\end{aligned}$$

But when n is even we know from Spottiswoode, as above, that $A_{rs} = -A_{sr}$; consequently we have in this case

$$\frac{\partial\Delta}{\partial a_{rs}} = 2A_{rs},$$

as Brioschi affirmed.* In the third place (§ 7, 5; pp. 28, 29) he applies Jacobi's general theorem

$$\begin{vmatrix} A_{rr} & A_{rs} \\ A_{sr} & A_{ss} \end{vmatrix} = \Delta\frac{\partial^2\Delta}{\partial a_{rr}\partial a_{ss}},$$

* It ought to be noticed also that Baltzer uses the equation

$$\frac{\partial\Delta}{\partial a_{rs}} = A_{rs} - A_{sr}$$

to verify Spottiswoode's theorem for the case where Δ is odd-ordered, the reasoning being that as Δ is then known to be zero, so also must $\partial\Delta/\partial a_{rs}$, and that therefore $A_{rs} = A_{sr}$.

as Brioschi did, to the case where Δ is zero-axial skew and of odd order to obtain the result

$$A^2_{rs} = A_{rr} \cdot A_{ss};$$

and he takes the further step of deducing from it the result

$$A_{r1} : A_{r2} : A_{r3} : \ldots . = \sqrt{A_{11}} : \sqrt{A_{22}} : \sqrt{A_{33}} : \ldots$$

thus showing, as he says (1) that the ratios on the left are independent of r, and (2) that, when the sign of one of the roots has been fixed, the others are known ("dass durch das Zeichen einer unter diesen Wurzeln die Zeichen der übrigen Wurzeln bestimmt sind.")

Turning now to the section specially set apart for the consideration of skew determinants, we find that it opens with Cayley's theorem regarding a zero-axial determinant of even order, the requirement being, as here worded, to prove that such a determinant is the square of *a rational integral function of the elements.* The proof is essentially the same as Spottiswoode's and Brioschi's, and differs from Cayley's merely in that it does not begin with a determinant of a more general form than is necessary,—a point which it is desirable to insist upon, as Baltzer ignores the fact, and then does not hesitate to say in a footnote that Cayley's proof "leaves manifold doubts unrelieved." In fact the theorem which Cayley proves is, that *if a zero-axial skew determinant of odd order be 'bordered' the resulting determinant is the product of two Pfaffians:* whereas what the three others prove, is the particular case of this in which the skewness extends to the bordering elements.

The development with which the proof begins Baltzer writes in the form

$$\Delta = a_{11}A_{11} - \sum_{rs} a_{r1}a_{1s}A'_{rs},$$

where A′ is the cofactor of a_{rs} in A_{11}, and r and s have the values 2, 3, . . . , n. He then uses the fact that A_{11} is a zero-axial skew determinant of odd order, and that therefore by a preceding result

$$A'_{rs} = A'_{sr} = \sqrt{A'_{rr}\,A'_{ss}};$$

so that there is obtained

$$\Delta = \sum_{rs} a_{1r}\, a_{1s} \sqrt{A'_{rr}\, A'_{ss}}\,;$$

and since in this aggregate the values possible for r are exactly those possible for s, he concludes (without knowing the signs of the terms of the aggregate, be it observed) that it is resolvable into two factors, viz.

$$\Big(\sum_r a_{1r} \sqrt{A'_{rr}}\Big)\Big(\sum_s a_{1s} \sqrt{A'_{ss}}\Big).$$

It is then argued that the two factors are identical even in the signs of their various terms "da durch das Zeichen einer Wurzel die Zeichen der übrigen bestimmt sind"; and that therefore

$$\Delta = \Big(\sum_r a_{1r} \sqrt{A'_{rr}}\Big)^2,$$

$$\text{and} \quad \sqrt{\Delta} = \sum_r a_{1r} \sqrt{A'_{rr}},$$

– an aggregate of $n-1$ terms, since the values to be given to r are $2, 3, \ldots, n$. The next step consists in pointing out that A'_{rr} being a determinant similar to Δ but of order $n-2$, it must follow that $\sqrt{A'_{rr}}$ can in the same way be expressed as an aggregate of $n-3$ terms, and that this process can be continued until the minor under the root-sign is of the 2nd order, when manifestly its value is the square of one of its elements. The final result thus is that $\sqrt{\Delta}$ is expressible as an aggregate of $(n-1)(n-3)\ldots 3.1$ terms each of which is the product of $\frac{1}{2}n$ elements whose collected suffixes form a permutation of $1, 2, \ldots, n$.

By way of corollary to this it is pointed out that

$$\pm a_{12}\, a_{34} \ldots a_{n-1,\,n}$$

is one of the terms of the aggregate, and the same is proved by showing that the square of this is a term of Δ, the reasoning being as follows:—Since in every case $a_{rs} = -a_{sr}$ we have

$$\begin{aligned}(a_{12}a_{34} \cdots a_{n-1,\,n})^2 &= (a_{12}a_{34} \cdots a_{n-1,n}) \cdot (-)^{\frac{1}{2}n}(a_{21}a_{43} \cdots a_{n,\,n-1}),\\ \text{and } \therefore \quad &= (-)^{\frac{1}{2}n}(a_{12}\, a_{21}\, a_{34}\, a_{43} \cdots a_{n-1,\,n}\, a_{n,\,n-1}),\end{aligned}$$

which clearly contains n elements, one from every row and one

from every column of Δ, and will therefore be a term of Δ if only we can show that the number of inversions of order in

$$2,1,\ 4,3,\ 6,5,\ \ldots,\ n,\ n-1$$

is $\frac{1}{2}n$, a fact which is self-evident.

Baltzer's proof that the rational integral function H, which is the square root of Δ, changes signs when two suffixes, r and s, are interchanged is a simplification of Brioschi's, the operation and even the notion of differentiation being dispensed with. The function resulting from the change being H′ he concludes like Brioschi that

$$\mathrm{H}^2 = \mathrm{H}'^2\,;$$

also the aggregate of the terms in H which contain a_{rs} being $a_{rs}\,\mathrm{B}$, say, he infers as Brioschi does that B cannot be affected by the change, and that therefore $a_{rs}\,\mathrm{B}$ will be altered into $a_{sr}\,\mathrm{B}$ or $-a_{rs}\,\mathrm{B}$. Here, however, he brings the demonstration quickly to a satisfactory end by saying that since some of the terms of H′ are thus seen to differ in sign only from the corresponding terms of H, the equation $\mathrm{H}^2 = \mathrm{H}'^2$ shows all of them must so differ; and this is what was to be proved.

Jacobi's notation for the function H is then introduced, the formal intimation being that (1, 2, 3, . . . , n) *is used to denote the aggregate whose first term is* $a_{12}a_{34}\ \ldots\ a_{n-1,\,n}$ *and whose square is* Δ. The other value of $\sqrt{\Delta}$ is thus of course representable by (2, 1, 3, . . . , n), (2, 3, . . . n, 1), or . . . As this implies also that

$$\sqrt{\mathrm{A}'_{rr}} = \pm(2,\ 3,\ \ldots,\ r-1,\ r+1,\ \ldots,\ n)$$

we have now the means, so far as symbolism is concerned, of removing the ambiguity from the various terms of the identity

$$\sqrt{\Delta} = a_{12}\sqrt{\mathrm{A}'_{22}} + a_{13}\sqrt{\mathrm{A}'_{33}} + \ \ldots\ + a_{1n}\sqrt{\mathrm{A}'_{nn}}.$$

As for the knowledge necessary to use the symbolism aright, Baltzer's dictum is *that the sign taken to precede* (2,3, . . . ,$r-1$, $r+1$, . . . ,n) *in substituting for* $\sqrt{\mathrm{A}'_{rr}}$ *must be such that the equation*

$$\sqrt{\mathrm{A}'_{rr}}\cdot\sqrt{\mathrm{A}'_{ss}} = \mathrm{A}'_{rs}$$

will be satisfied; and this he proves will take place when the

sign-factor of $(2,3, \ldots, r-1, r+1, \ldots, n)$ is $(-1)^r$. By hypothesis, he says, the left-hand side

$$=(-1)^r(2,3,\ldots,r-1,r+1,\ldots,n)\cdot(-1)^s(2,3,\ldots,s-1,s+1,\ldots,n),$$
$$=(-1)^{r+s}(2,3,\ldots,r-1,r+1,\ldots,n)\,(2,3,\ldots,s-1,s+1,\ldots,n),$$

and therefore by a previous theorem

$$=-(-1)^{r+s}(2,3,\ldots,r-1,r+1,\ldots,n)\,(3,\ldots,s-1,s+1,\ldots,n,2),$$

the first term of which is

$$-(-1)^{r+s}a_{23}\ \ldots\ a_{n-1,n}\cdot a_{34}\ \ldots\ a_{n,2},$$

or

$$-(-1)^{r+s}a_{23}\,a_{34}\ \ldots\ a_{n-1,n}a_{n2}\,;$$

and the right-hand side

$$=\text{cofactor of } a_{rs} \text{ in } \begin{vmatrix} a_{22} & a_{23} & \ldots & a_{2m} \\ a_{32} & a_{33} & \ldots & a_{3n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix},$$

$$=(-1)^{r+s}\begin{vmatrix} a_{22} & a_{23} & \ldots & a_{2,s-1} & a_{2,s+1} & \ldots & a_{2n} \\ a_{32} & a_{33} & \ldots & a_{3,s-1} & a_{3,s+1} & \ldots & a_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{r-1,2} & a_{r-1,3} & \ldots & a_{r-1,s-1} & a_{r-1,s+1} & \ldots & a_{r-1,n} \\ a_{r+1,2} & a_{r+1,3} & \ldots & a_{r+,s-1} & a_{r+1,s+1} & \ldots & a_{r+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n,2} & a_{n,3} & \ldots & a_{n,s-1} & a_{n,s+1} & \ldots & a_{n,n} \end{vmatrix},$$

and therefore, on account of the translation of the first column to the last place,

$$=-(-1)^{r+s}\begin{vmatrix} a_{23} & \ldots & a_{2,s-1} & a_{2,s+1} & \ldots & a_{2,n} & a_{22} \\ a_{33} & \ldots & a_{3,s-1} & a_{3,s+1} & \ldots & a_{3,n} & a_{32} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{r-1,3} & \ldots & a_{r-1,s-1} & a_{r-1,s+1} & \ldots & a_{r-1,n} & a_{r-1,2} \\ a_{r+1,3} & \ldots & a_{r+1,s-1} & a_{r+1,s+1} & \ldots & a_{r+1,n} & a_{r+,2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n,3} & \ldots & a_{n,s-1} & a_{n,s+1} & \ldots & a_{n,n} & a_{n,2} \end{vmatrix},$$

the first term of which is

$$-(-1)^{r+s}\,a_{23}\,a_{34}\ldots a_{n-1,n}\,a_{n,2}\,,$$

exactly as before.

To the proof no note is appended drawing attention to the fact that the very same result would have been reached by taking $(-1)^{r-1}$, or indeed $(-1)^{r-t}$, instead of $(-1)^r$ for the sign-factor of $(2,3,\ldots, r-1, r+1,\ldots, n)$.

The very next step taken in accordance with the above mentioned dictum, is to make the substitution in the right-hand side of the equation

$$\sqrt{\Delta} = a_{12}\sqrt{A'_{22}} + a_{13}\sqrt{A'_{33}} + \ldots + a_{1n}\sqrt{A'_{nn}},$$

the first term being used to decide whether $(1,2,3, \ldots, n)$ or $-(1,2,3, \ldots, n)$ has to be substituted for the left-hand side, and the final result being

$$(1,2,3,\ldots,n) = a_{12}(3,\ldots,n) + a_{13}(4,\ldots,n,2) + \ldots + a_{1n}(2,\ldots,n-1).$$

Since $(3,4, \ldots, n)$ is the cofactor of a_{12} in $(1,2,3, \ldots, n)$ and the differential-quotient of the latter with respect to a_{12} is the same, it immediately follows from this that

$$\sqrt{\Delta} = a_{12}\frac{\partial\sqrt{\Delta}}{\partial a_{12}} + a_{13}\frac{\partial\sqrt{\Delta}}{\partial a_{13}} + \ldots + a_{1n}\frac{\partial\sqrt{\Delta}}{\partial a_{1n}}.$$

Baltzer, however, obtains a more general result by going back to the corresponding more general theorem in determinants, viz., the theorem

$$\Delta = a_{r1}A_{r1} + a_{r2}A_{r2} + \ldots + a_{rn}A_{rn},$$

with which he associates

$$0 = a_{r1}A_{s1} + a_{r2}A_{s2} + \ldots + a_{rn}A_{sn};$$

substituting $\sqrt{\Delta}\dfrac{\partial\sqrt{\Delta}}{\partial a_{rs}}$ for A_{rs}; and then dividing both sides $\sqrt{\Delta}$. In the results,

$$\sqrt{\Delta} = a_{r1}\frac{\partial\sqrt{\Delta}}{\partial a_{r1}} + \ldots + a_{rn}\frac{\partial\sqrt{\Delta}}{\partial a_{rn}},$$

$$0 = a_{r1}\frac{\partial\sqrt{\Delta}}{\partial a_{s1}} + \ldots + a_{rn}\frac{\partial\sqrt{\Delta}}{\partial a_{sn}},$$

it has to be noticed that there is no term in $\partial\sqrt{\Delta}/\partial a_{rr}$.

By comparison of the first of these with the immediately preceding result (the recurring law of development), he deduces the quite general identity regarding the two forms of the cofactor of

a_{rs} in $\surd\Delta$—the identity, that is to say, with which we were inclined to start.

His words are—

"Setzt man

$$\begin{aligned}\surd\Delta &= (r,1,2,\ldots, r-1,r+1\ldots,n)\\ &= a_{r1}(2,\ldots,n) + a_{r2}(3,\ldots,n,1) + \ldots\end{aligned}$$

so findet man

$$\frac{\partial\surd\Delta}{\partial a_{rs}} = (s+1,\ldots,n,1,\ldots,s-1),$$

in welchem Cyclus die Suffixe r and s fehlen."

In regard to this the reader has, of course, to note that $(r,1,2,\ldots, r-1,r+1\ldots,n)$ being only one of the two values of $\surd\Delta$, the differential-quotient obtained is also only one of two; in other words, that the result reached is really

$$\partial(r,1,2,\ldots, r-1, r+1,\ldots,n)/\partial a_{rs} = (s+1,\ldots,n,1,\ldots, s-1),$$

where from 1 to $s-1$ and from $s+1$ to n the integers appear in natural order, save that r is omitted.

The remainder of the chapter or section, which contains no new feature, refers to Cayley's expansion of a determinant arranged according to products of elements of the principal diagonal, and the application of this to skew determinants whose diagonal elements are each equal to z.

On the Motion produced in an Infinite Elastic Solid by the Motion through the Space occupied by it of a body acting on it only by Attraction or Repulsion. By Lord Kelvin.

(Read July 16, 1900.)

§ 1. The title of the present communication describes a pure problem of abstract mathematical dynamics, without indication of any idea of a physical application. For a merely mathematical journal it might be suitable, because the dynamical subject is certainly interesting both in itself and in its relation to waves and vibrations. My reason for occupying myself with it, and for offering it to the Royal Society of Edinburgh, is that it suggests a conceivable explanation of the greatest difficulty hitherto presented by the undulatory theory of light; the motion of ponderable bodies through infinite space occupied by an elastic solid.*

§ 2. In consideration of the confessed object, and for brevity, I shall use the word atom to denote an ideal substance occupying a given portion of solid space, and acting on the ether within it and around it, according to the old-fashioned eighteenth century idea of attraction and repulsion. That is to say, every infinitesimal volume A of the atom acts on every infinitesimal volume B of the ether with a force in the line PQ joining the centres of these two volumes, equal to

$$Af(P, PQ)\rho B \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where ρ denotes the density of the ether at Q, and f (P, PQ) denotes a quantity depending on the position of P and on the

* The so-called "electro-magnetic theory of light" does not cut away this foundation from the old undulatory theory of light. It adds to that primary theory an enormous province of transcendent interest and importance; it demands of us not merely an explanation of all the phenomena of light and radiant heat by transverse vibrations of an elastic solid called ether, but also the inclusion of electric currents, of the permanent magnetism of steel and lodestone, of magnetic force, and of electrostatic force, in a comprehensive ethereal dynamics.

distance PQ. The whole force exerted by the atom on the portion ρB of the ether at Q, is the resultant of all the forces calculated according to (1), for all the infinitesimal portions A into which we imagine the whole volume of the atom to be divided.

§ 3. According to the doctrine of the potential in the well-known mathematical theory of attraction, we find rectangular components of this resultant as follows:—

$$\left.\begin{aligned} X = \rho B\frac{d}{dx}\phi(x, y, z)\,; \quad Y = \rho B\frac{d}{dy}\phi(x, y, z)\,; \\ Z = \rho B\,\frac{d}{dz}\,\phi(x, y, z) \end{aligned}\right\} \quad . \ . \ (2),$$

where x, y, z denote co-ordinates of Q referred to lines fixed with reference to the atom, and ϕ denotes a function (which we call the potential at Q due to the atom) found by summation as follows:—

$$\phi = \iiint A \int_{PQ}^{\infty} dr\, f(P, r) \quad . \ . \ . \ . \ . \quad (3),$$

where $\iiint$A denotes integration throughout the volume of the atom.

§ 4. The notation of (1) has been introduced to signify that no limitation as to admissible law of force is essential; but no generality that seems to me at present practically desirable, is lost if we assume, henceforth, that it is the Newtonian law of the inverse square of the distance. This makes

$$f(P, PQ) = \frac{a}{PQ^2} \quad . \ . \ . \ . \ . \ . \ . \quad (4),$$

and therefore

$$\int_{PQ}^{\infty} dr\, f(P, r) = \frac{a}{PQ} \quad . \ . \ . \ . \ . \ . \ . \quad (5),$$

where a is a coefficient specifying for the point, P, of the atom, the intensity of its attractive quality for ether. Using (5) in (3) we find

$$\phi = \iiint A\,\frac{a}{PQ} \quad . \ . \ . \ . \ . \ . \quad (6),$$

and the components of the resultant force are still expressed by (2). We may suppose a to be either positive or negative (positive for attraction and negative for repulsion); and in fact in our first

and simplest illustration of the problem we suppose it to be positive in some parts and negative in other parts of the atom, in such quantities as to fulfil the condition

$$\iiint A\alpha = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

§ 5. As a first and very simple illustration, suppose the atom to be spherical, of radius unity, with concentric interior spherical surfaces of equal density. This gives, for the direction of the resultant force on any particle of the ether, whether inside or outside the spherical boundary of the atom, a line through the centre of the atom. The further assumption of (7) may now be expressed by

$$\int_0^1 dr\, r^2 \alpha = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8);$$

and this, as we are now supposing the forces between every particle of the atom and every particle of the ether to be subject to the Newtonian law, implies, that the resultant of its attractions and repulsions is zero for every particle of ether outside the boundary of the atom. To simplify the case to the utmost, we shall further suppose the distribution of positive and negative density of the atom, and the law of compressibility of the ether, to be such, that the average density of the ether within the atom is equal to the undisturbed density of the ether outside. Thus the attractions and repulsions of the atom in lines through its centre produce, at different distances from its centre, condensations and rarefactions of the ether, with no change of the total quantity of it within the boundary of the atom; and therefore produce no disturbance of the ether outside. To fix the ideas, and to illustrate the application of the suggested hypothesis to explain the refractivity of ordinary isotropic transparent bodies such as water or glass, I have chosen a definite particular case in which the distribution of the ether when at rest within the atom is expressed by the following formula, and partially shown in the accompanying diagram, and tables of calculated numbers:—

$$r^3 = \frac{r'^3}{1 + K(1 - r')^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Here, r' denotes the undisturbed distance from the centre of the atom, of a particle of the ether which is at distance

r when at rest under the influence of the attractive and repulsive forces. According to this notation $\frac{4\pi}{3}\delta(r^3)$ is the disturbed volume of a spherical shell of ether whose undisturbed radius is r' and thickness $\delta r'$ and volume $\frac{4\pi}{3}\delta(r'^3)$. Hence, if we denote the disturbed and undisturbed densities of the ether by ρ and unity respectively, we have

$$\rho\delta(r^3) = \delta(r'^3) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

This, with (9), gives

$$\rho = \frac{3[1 + \mathrm{K}(1-r')^2]^2}{3 + \mathrm{K}(3-r')(1-r')} \quad . \quad . \quad . \quad . \quad . \quad (11).$$

This gives 1+K for the density of the ether at the centre of the atom. In order that the disturbance may suffice for refractivities such as those of air, or other gases, or water, or glass, or other transparent liquids or isotropic solids, according to the dynamical theory explained in § (16) below, I find that K may for some cases be about equal to 100, and for others must be considerably greater. I have therefore taken K = 100, and calculated and drawn the accompanying tables and diagram accordingly.

Table I.

Col. 1.	Col. 2.	Col. 3.	Col. 3′.	Col. 4.	Col. 5.
r'.	$\frac{r'^3}{r^3} = 1 + \mathrm{K}(1-r')^2$.	r.	$r'-r$.	ρ.	$(\rho-1)r^2$.
0·00	101·0	0·000	0·000	101·0	0·000
·05	91·25	·011	·039	88·1	·011
·10	82·0	·023	·077	75·3	·039
·20	65·0	·049	·151	55·8	·132
·30	50·0	·082	·218	39·1	·256
·40	37·0	·120	·280	25·8	·357
·50	26·0	·169	·331	15·8	·423
·60	17·0	·233	·367	8·76	·423
·70	10·0	·325	·375	4·17	·338
·80	5·0	·468	·332	1·60	·131
·85	3·25	·578	·272	0·90	−0·033
·90	2·00	·715	·185	0·50	− ·256
·95	1·25	·865	·085	·35	− ·486
·96	1·16	·897	·063	·36	− ·515
·97	1·09	·928	·042	·39	− ·525
·98	1·04	·957	·023	·46	− ·495
·99	1·01	·982	·008	·61	− ·376
1·00	1·00	1·000	·000	1·00	− ·000

TABLE II.

Col. 1.	Col. 2.	Col. 3.	Col. 4.	Col. 5.
r.	r'.	$r-r'$.	ρ.	$(\rho-1)r^2$.
0·00	0·000	0·000	101·00	0·000
·02	·091	·071	78·5	·030
·04	·169	·129	64·4	·191
·06	·235	·175	49·6	·175
·08	·297	·217	39·5	·246
·10	·351	·251	31·8	·308
·20	·551	·351	11·8	·432
·30	·677	·377	5·00	·360
·40	·758	·358	2·46	·234
·50	·816	·316	1·34	·085
·60	·858	·258	0·82	−0·065
·70	·895	·195	0·53	− ·231
·80	·929	·129	0·38	− ·397
·90	·961	·061	0·36	− ·518
1·00	1·000	·000	1·00	·000

§ 6. The diagram (fig. 1) helps us to understand the displacement of ether and the resulting distribution of density, within the atom. The circular arc marked 1·00 indicates a spherical portion of the boundary of the atom; the shorter of the circular arcs marked ·95, ·90, ... ·20, ·10 indicate spherical surfaces of undisturbed ether of radii equal to these numbers. The position of the spherical surfaces of the same portions of ether under the influence of the atom, are indicated by the arc marked 1·00, and the longer of the arcs marked ·95, ·90, . . . ·50, and the complete circles marked ·40, ·30, ·20, ·10. It may be remarked that the average density of the ether within any one of the disturbed spherical surfaces, is equal to the cube of the ratio of the undisturbed radius to the disturbed radius, and is shown numerically in column 2 of Table I. Thus, for example, looking at the table and diagram, we see that the cube of the radius of the short arc marked ·50 is 26 times the cube of the radius of the long arc marked ·50, and therefore the average density of the ether within the spherical surface corresponding to the latter is 26 times the density (unity) of the undisturbed ether within the spherical surface corresponding to the former. The densities shown in column 4 of each table are the

densities of the ether at (not the average density of the ether within) the concentric spherical surfaces of radius r in the atom. Column 5 in each table shows $1/4\pi e$ of the excess (positive or negative) of the quantity of ether in a shell of radius r and infinitely small thickness e as disturbed by the atom above the quantity in a shell of the same dimensions of undisturbed ether. The formula of col. 2 makes $r=1$ when $r'=1$; that is to say, the total quantity of the disturbed ether within the radius of the atom is the same as that of undisturbed ether in a sphere of the same radius. Hence the sum of the quantities of ether calculated from col. 5 for consecutive values of r, with infinitely small differences from $r=0$ to $r=1$, must be zero. Without calculating for smaller differences of r than those shown in either of the tables, we find a close verification of this result by drawing, as in fig. 2, a curve to represent $(\rho-1)r^2$ through the points for which the value is given in one or other of the tables, and measuring

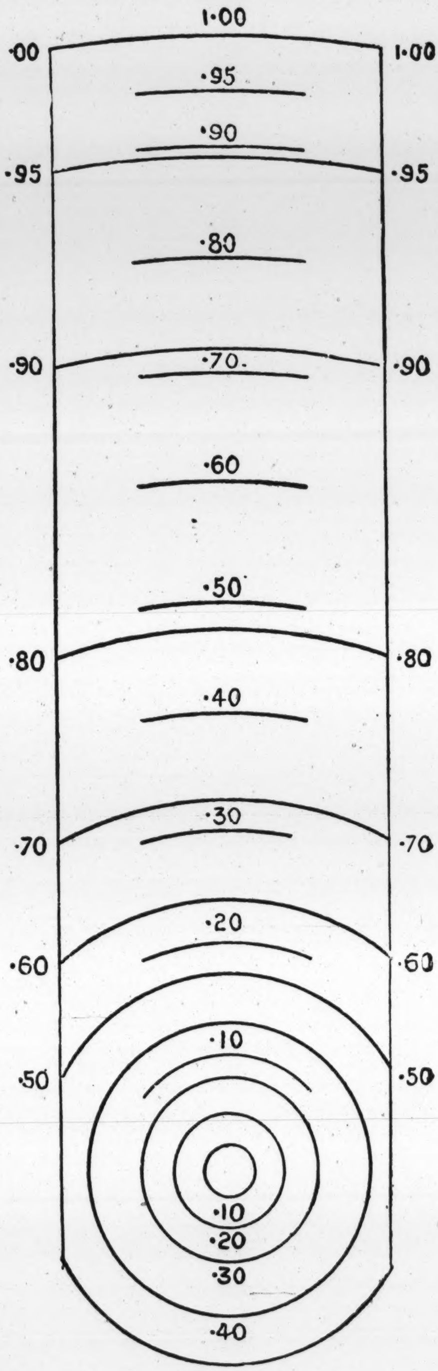


Fig. 1.

the areas on the positive and negative sides of the line of

abscissas. By drawing on paper (four times the scale of the annexed diagram), showing engraved squares of ·5 inch and ·1 inch, and counting the smallest squares and parts of squares in the two areas, I have verified that they are equal within less than 1 per cent. of either sum, which is as close as can be expected from the numerical approximation shown in the tables and from the accuracy attained in the drawing.

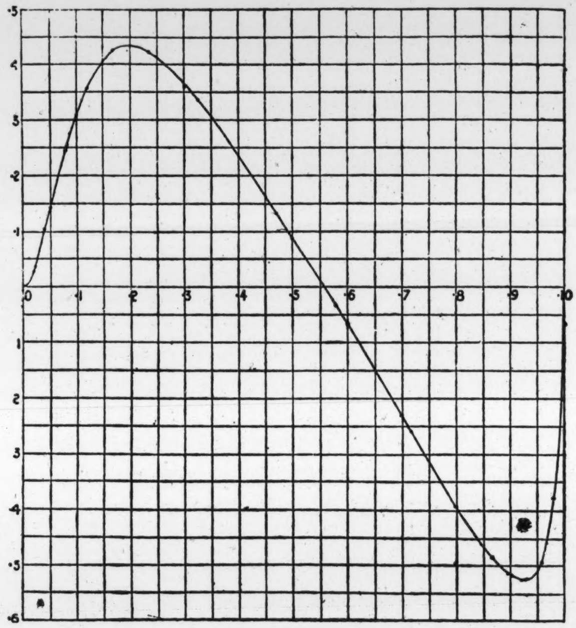


FIG. 2.

§ 7. In Table I. (argument r') all the quantities are shown for chosen values of r', and in Table II. for chosen values of r. The calculations for Table I. are purely algebraic, involving merely cube roots beyond elementary arithmetic. To calculate in terms of given values of r the results shown in Table II. involves the solution of a cubic equation. They have been actually found by aid of a curve drawn from the numbers of col. 3, Table I., showing r in terms of r'. The numbers in col. 2 of Table II. showing, for chosen values of r, the corresponding values of r', have been taken from the curve; and we may verify that they are approximately equal to the roots of the equation shown at the head of col. 2 of Table I., regarded as a cubic for r' with any given values of r and K.

Thus, for example, taking $r' = \cdot 929$ we calculate $r = \cdot 811$,
" $r' = \cdot 816$ " $r = \cdot 498$,
" $r' = \cdot 677$ " $r = \cdot 301$,
" $r' = \cdot 091$ " $r = \cdot 0208$,

where we should have $r = \cdot 8$, $\cdot 5$, $\cdot 3$, and $\cdot 02$ respectively. These approximations are good enough for our present purpose.

§ 8. The diagram of fig. 2 is interesting, as showing how, with densities of ether varying through the wide range of from ·35 to 101, the whole mass within the atom is distributed among the concentric spherical surfaces of equal density. We see by it, interpreted in conjunction with col. 4 of the tables, that from the centre to ·56 of the radius the density falls from 101 to 1. For radii from ·56 to 1, the values of $(\rho - 1)r^2$ decrease to a negative minimum of ·525 at $r = \cdot 93$, and rise to zero at $r = 1$. The place of minimum density is of course inside the radius at which $(\rho - 1)r^2$ is a minimum; by cols. 4 and 3 of Table I., and cols. 4 and 1 of Table II., we see that the minimum density is about ·35, and at distance approximately ·87 from the centre.

§ 9. Let us suppose now our atom to be set in motion through space occupied by ether, and kept in motion with a uniform velocity v, which we shall first suppose to be infinitely small in comparison with the propagational velocity of equivoluminal* waves through pure ether undisturbed by any other substance than that of the atom. The velocity of the earth in its orbit round the sun being about 1/10,000 of the velocity of light, is small enough to give results, kinematic and dynamic, in respect to the relative motion of ether and the atoms constituting the earth closely in agreement with this supposition. According to it, the position of every particle of the ether at any instant is the same as if the atom were at rest; and to find the motion produced in the ether by the motion of the atom, we have a purely kinematic problem of which an easy graphic solution is found by marking on a diagram the successive positions thus determined for any particle of the ether, according to the positions

* That is to say, waves of transverse vibration, being the only kind of wave in an isotropic solid in which every part of the solid keeps its volume unchanged during the motion. See *Phil. Mag.*, May, August, and October 1899.

of the atom at successive times with short enough intervals between them, to show clearly the path and the varying velocity of the particle.

§ 10. Look, for example, at fig. 3, in which a semi-circumference of the atom at the middle instant of the time we are going to consider, is indicated by a semi-circle $C_{20}AC_0$, with diameter C_0C_{20} equal to two units of length. Suppose the centre of the atom to move from right to left in the straight line C_0C_{20} with velocity ·1, taking for unit of time the time of travelling 1/10 of the radius. Thus, reckoning from the time when the centre is at C_0, the times when it is at C_2, C_5, C_{10}, C_{18}, C_{20} are 2, 5, 10, 18, 20. Let Q' be the undisturbed position of a particle of ether before time 2 when the atom reaches it, and after time 18 when the atom leaves it. This implies that $Q'C_2=Q'C_{18}=1$, and $C_2C_{10}=C_{10}C_{18}=\cdot 8$, and therefore $C_{10}Q'=\cdot 6$. The position of the particle of ether, which when undisturbed is at Q, is found for any instant t of the disturbance as follows:—

Take $C_0C=t/10$; draw $Q'C$, and calling this r' find $r'-r$ by formula (9), or Table I. or II.: in $Q'C$ take $Q'Q=r'-r$. Q is the position at time t of the particle whose undisturbed position is Q'. The drawing shows the construction for $t=5$. The positions at times 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18 are indicated by the dots marked 2, 3, 4, 5, 6, 7, 8, 9, 0, 1, 2, 3, 4, 5, 6, 7, 8 on the closed curve with a corner at Q', which has been found by tracing a smooth curve through them. This curve, which, for brevity, we shall call the orbit of the particle, is clearly tangential to the lines $Q'C_2$ and $Q'C_{18}$. By looking to the formula (9), we see that the velocity of the particle is zero at the instants of leaving Q' and returning to it. Fig. 4 shows the particular orbit of fig. 3, and nine others drawn by the same method; in all ten orbits of ten particles whose undisturbed positions are in one line at right angles to the line of motion of the centre of the atom, and at distances 0, ·1, ·2, . . . ·9 from it. All these particles are again in one straight line at time 10, being what we may call the time of mid-orbit of each particle. The numbers marked on the right-hand halves of the orbits are times from the zero of our reckoning; the numbers 1, 2, 3 . . . etc. on the left correspond to times 11, 12, 13 . . . of our reckoning as

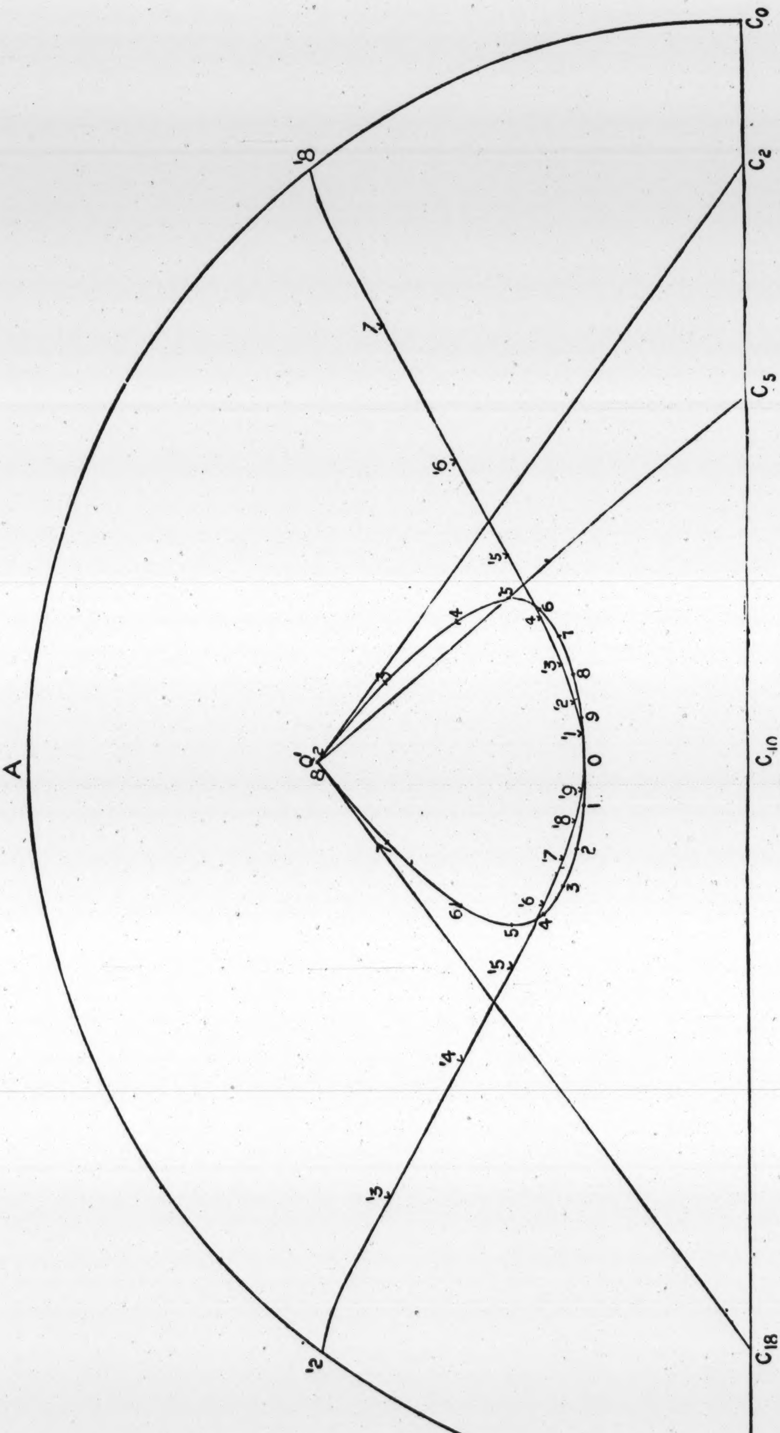


FIG. 3.

hitherto, or to times 1, 2, 3 . . . after mid-orbit passages. Lines drawn across the orbits through 1, 2, 3 . . . on the left, show simultaneous positions of the ten particles at times 1, 2, 3 after mid-

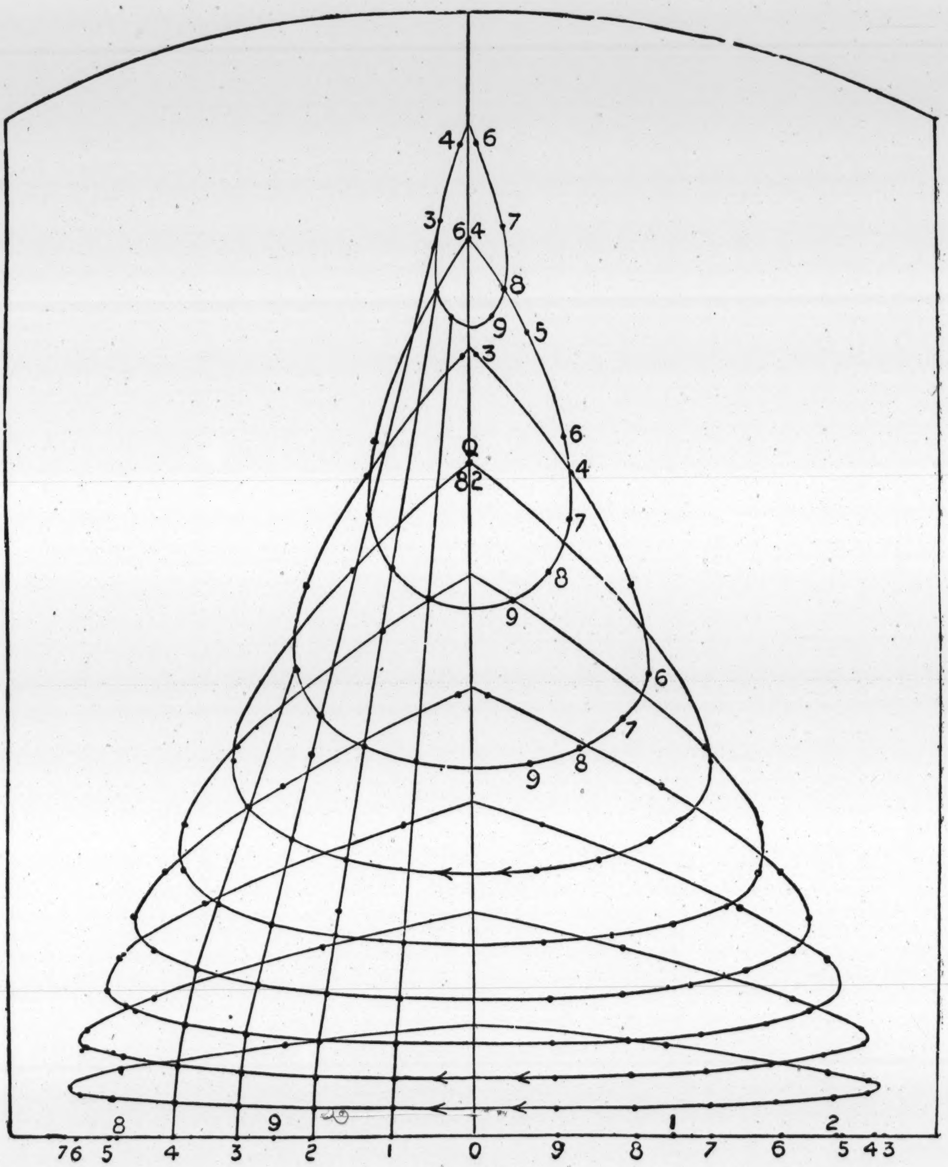


FIG. 4.

orbit. The line drawn from 4 across seven of the curved orbits, shows for time 4 after mid-orbit, simultaneous positions of eight particles, whose undisturbed distances are 0, ·1, . . . ·7. Remark that the orbit for the first of these ten particles is a straight line.

§ 11. We have thus in § 10 solved one of the two chief kinematic questions presented by our problem:—to find the orbit of a particle of ether as disturbed by the moving atom, relatively to the surrounding ether supposed fixed. The other question, to find the path traced through the atom supposed fixed while, through all space outside the atom, the ether is supposed to move uniformly in parallel lines, is easily solved, as follows:—Going back to fig. 3, suppose now that instead of, as in § 10, the atom moving from right to left with velocity ·1 and the ether outside it at rest, the atom is at rest and the ether outside it is moving from left to right with velocity ·1. Let '2, '3, '4, '5, '6, '7, '8, '9, 0, '1, '2, '3, '4, '5, '6, '7, '8 be the path of a particle of ether through the atom marked by seventeen points corresponding to the same numbers unaccented showing the orbit of the same particle of ether on the former supposition. On both suppositions, the position of the particle of ether at time 10 from our original era (§ 10), is marked 0. For times 11, 12, 13, etc., the positions of the particle on the former supposition are marked 1, 2, 3, 4, 5, 6, 7, 8 on the left half of the orbit. The positions of the same particle on the present supposition are found by drawing from the points 1, 2, 3, . . . 7, 8 parallel lines to the right, 1 '1, 2 '2, 3 '3, . . . 7 '7, 8 '8, equal respectively to ·1, ·2, ·3, . . . ·7, ·8 of the radius of the atom, being our unit of length. Thus we have the latter half of the passage of the particle through the atom; the first half is equal and similar on the left-hand side of the atom. Applying the same process to every one of the ten orbits shown in fig. 4, and to the nine orbits of particles whose undisturbed distances from the central line on the other side are ·1, ·2, . . . ·9, we find the set of stream-lines shown in fig. 5. The dots on these lines show the positions of the particles at times 0, 1, 2, . . . 19, 20 of our original reckoning (§ 10). The numbers on the stream-line of the particle whose undisturbed distance from the central line is ·6 are marked for comparison with fig. 3. The lines drawn across the stream-lines on the left-hand side of fig. 5, show simultaneous positions of rows of particles of ether which, when undisturbed, are in straight lines perpendicular to the direction of motion. The quadrilaterals thus formed within the left-hand semicircle show the

figures to which the squares of ether, seen entering from the left-hand end of the diagram, become altered in passing through

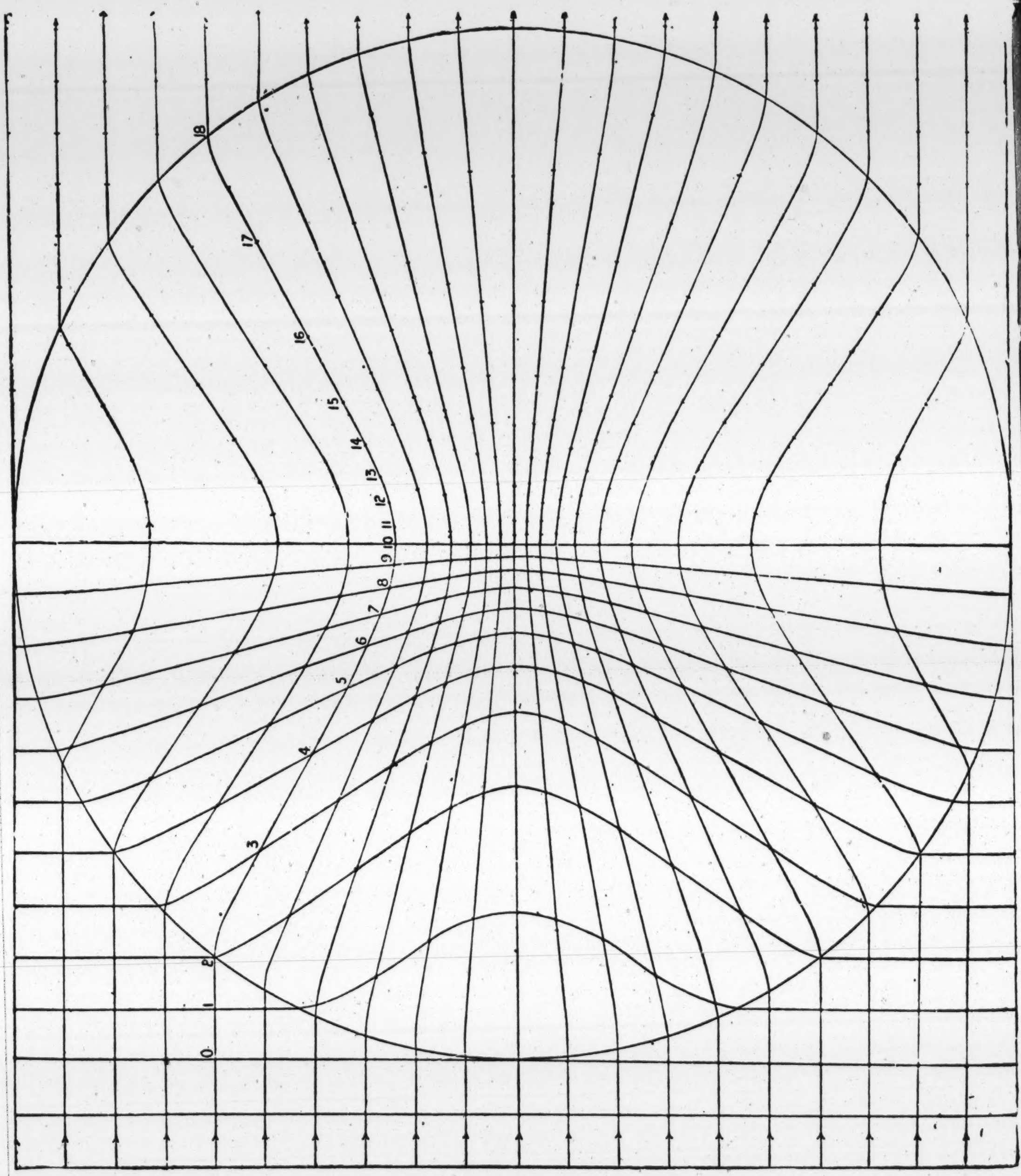


FIG. 5.

the atom. Thus we have completed the solution of our second chief kinematic question.

§ 12. The first dynamic question that occurs to us, returning to the supposition of moving atom and of ether outside it at rest, is:—What is the total kinetic energy (κ) of the portion of the ether which at any instant is within the atom? To answer it, think of an infinite circular cylinder of the ether in the space traversed by the atom. The time-integral from any era $t=0$ of the total kinetic energy of the ether in this cylinder is $t\kappa$; because the ether outside the cylinder is undisturbed by the motion of the atom according to our present assumptions. Consider any circular disk of this cylinder of infinitely small thickness e. After the atom has passed it, it has contributed to $t\kappa$, an amount equal to the time-integral of the kinetic energies of all the orbits of small parts into which we may suppose it divided, and it contributes no more in subsequent time. Imagine the disk divided into concentric rings of rectangular cross-section $e\,dr'$. The mass of one of these rings is $2\pi r'\,dr'\,e$ because its density is unity; and all its parts move in equal and similar orbits. Thus we find that the total contribution of the disk amounts to

$$2\pi e\int_0^1 dr'\,r'\int ds^2/dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

where $\int ds^2/dt$ denotes integration over one-half the orbit of a particle of ether whose undisturbed distance from the central line is r' (because $\frac{1}{2}ds^2/dt^2$ is the kinetic energy of an ideal particle of unit mass moving in the orbit considered). Hence the time-integral κt is wholly made up by contributions of successive disks of the cylinder. Hence (12) shows the contribution per time e/q, q being the velocity of the atom; and (κ being the contribution per unit of time) we therefore have

$$\kappa = 2\pi q\int_0^1 dr'\,r'\int ds^2/dt \quad . \quad . \quad . \quad . \quad . \quad (13).$$

§ 13. The double integral shown in (13) has been evaluated with amply sufficient accuracy for our present purpose by seemingly rough summations; firstly, the summations $\int ds^2/dt$ for the ten orbits shown in fig. 4, and secondly, summation of these sums each multiplied by $dr'\,r'$. In the summations for each half-orbit, ds has been taken as the lengths of the curve between the

consecutive points from which the curve has been traced. This implies taking $dt = 1$ throughout the three orbits corresponding to undisturbed distances from the central line equal respectively to 0, ·6, ·8, and throughout the other semi-orbits, except for the portions next the corner, which correspond essentially to intervals each <1. The plan followed is sufficiently illustrated by the accompanying Table III., which shows the whole process of calculating and summing the parts for the orbit corresponding to undisturbed distance ·7.

Table IV. shows the sums for the ten orbits and the products of each sum multiplied by the proper value of r', to prepare for the final integration, which has been performed by finding the area of a representative curve drawn on conveniently squared paper as described in § 6 above. The result thus found is ·02115. It is very satisfactory to see that, within ·1 per cent., this agrees with the simple sum of the widely different numbers shown in col. 3 of Table IV.

TABLE III.

Orbit $r' = {\cdot}7$.

ds.	ds^2.	dt.	ds^2/dt.
·006	·000036	0·14	·000257
·137	·018769	1·00	·018769
·112	·012544	1·00	·012544
·077	·005929	1·00	·005929
·050	·002500	1·00	·002500
·048	·002304	1·00	·002304
·050	·002500	1·00	·002500
·052	·002704	1·00	·002704
		Sum.........	·047507

TABLE IV.

r'.	$\int ds^2/dt$.	$\cdot 1 \times r' . \int ds^2/dt$.
·0	·0818	·00000
·1	·0804	·00080
·2	·0781	·00156
·3	·0769	·00231
·4	·0722	·00289
·5	·0670	·00335
·6	·0567	·00340
·7	·0475	·00332
·8	·0310	·00248
·9	·0114	·00102
	Sum.........	·02113

§ 14. Using in (13) the conclusion of § 13, and taking $q = 1$, we find

$$\kappa = 2\pi . {\cdot}002115 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14).$$

A convenient way of explaining this result is to remark that it is ·634 of the kinetic energy $\left(\frac{4\pi}{3} \cdot \frac{1}{2} ({\cdot}1)^2\right)$ of an ideal globe

of rigid matter of the same bulk as our atom, moving with the same velocity. Looking now at the definition of κ in the beginning of § 12, we may put our conclusion in words, thus:—The distribution of ethereal density within our ideal spherical atom represented by (11) with $K = 100$, gives rise to kinetic energy of the ether within it at any instant, when the atom is moving slowly through space filled with ether, equal to ·634 of the kinetic energy of motion with the same velocity through ideal void space, of an ideal rigid globe of the same bulk as the atom, and the same density as the undisturbed density of the ether. Thus if the atom, which we are supposing to be a constituent of real ponderable matter, has an inertia of its own equal to I per unit of its volume, the effective inertia of its motion through space occupied by ether will be $\frac{\pi}{6} s^3(I + \cdot 634)$; the diameter of the atom being now denoted by s (instead of 2 as hitherto), and the inertia of unit bulk of the ether being still (as hitherto) taken as unit of inertia. In all that follows we shall suppose I to be very great, much greater than 10^6; perhaps greater than 10^{12}.

§ 15. Consider now, as in § 11 above, our atom at rest; and the ether moving uniformly in the space around the atom, and through the space occupied by the atom, according to the curved stream-lines and the varying velocities shown in fig. 5. The effective inertia of any portion of the ether containing the atom will be greater than the simple inertia of an equal volume of the ether by the amount $\frac{\pi}{6} s^3 \cdot 634$. This follows from the well-known dynamical theorem that the total kinetic energy of any moving body or system of bodies is equal to the kinetic energy due to the motion of its centre of inertia, plus the sum of the kinetic energies of the motions of all its parts relative to the centre of inertia.

§ 16. Suppose now a transparent body—solid, liquid, or gaseous—to consist of an assemblage of atoms all of the same magnitude and quality as our ideal atom defined in § 2, and with I enormously great as described in § 14. The atoms may be all

motionless as in an absolutely cold solid, or they may have the thermal motions of the molecules of a solid, liquid, or gas at any temperature not so high but that the thermal velocities are everywhere small in comparison with the velocity of light. The effective inertia of the ether per unit volume of the assemblage will be exceedingly nearly the same as if the atoms were all absolutely fixed, and will therefore, by § 15, be equal to

$$1 + N\frac{\pi}{6}s^3 \cdot 634 \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

where N denotes the number of atoms per cubic centimetre of the assemblage, one centimetre being now our unit of length. Hence, if we denote by V the velocity of light in undisturbed ether, its velocity through the space occupied by the supposed assemblage of atoms will be

$$V/\left(1 + N\frac{\pi}{6}s^3 \cdot 634\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (16).$$

§ 17. For example, let us take $N = 4 \times 10^{20}$*; and, as I find suits the cases of oxygen and argon, $s = 1{\cdot}42 \times 10^{-8}$, which gives $N\frac{\pi}{6}s^3 = {\cdot}60 \times 10^{-3}$. The assemblage thus defined would, if condensed one-thousandfold, have ·6 of its whole volume occupied by the atoms and ·4 by undisturbed ether; which is somewhat denser than the cubic arrangement of globes (space unoccupied $= 1 - \frac{\pi}{6} = {\cdot}4764$), and less dense than the densest possible arrangement (space unoccupied $= 1 - \frac{\pi}{3\sqrt{2}} = {\cdot}2595$). Taking now $N\frac{\pi}{6}s^3 = {\cdot}60 \times 10^{-3}$ in (16), we find for the refractive index of our assemblage 1·00019, which is somewhat smaller than the refractive index of oxygen (1·000273). By taking

* I am forced to take this very large number instead of Maxwell's 19×10^{18}, as I have found it otherwise impossible to reconcile the known viscosities and the known condensations of hydrogen, oxygen, and nitrogen with Maxwell's theoretical formula $Ns^2 = \frac{1}{2\sqrt{2\pi}} \cdot \frac{v}{D} = {\cdot}3989\frac{v}{D}$, where v is the Newtonian velocity of sound in the particular gas, and D is its diffusivity, that is, its viscosity divided by its density. It must be remembered that Avogadro's law makes N the same for all gases.

a larger value than 100 in (11), we could readily fit the formula to give, in an assemblage in which $\cdot 6 \times 10^{-3}$ of the whole space is occupied by the atom, exactly the refractive index of oxygen, nitrogen, or argon, or any other gas. It is remarkable that according to the particular assumptions specified in § 5, a density of ether in the centre of the atom considerably greater than 100 times the density of undisturbed ether is required to make the refractivity as great as that of oxygen. There is, however, no difficulty in admitting so great a condensation of ether by the atom, if we are to regard our present problem as the basis of a physical hypothesis worthy of consideration.

§ 18. There is, however, one serious, perhaps insuperable, difficulty to which I must refer in conclusion: the reconciliation of our hypothesis with the result that ether in the earth's atmosphere is motionless relatively to the earth, seemingly proved by an admirable experiment designed by Michelson, and carried out with most searching care to secure a trustworthy result, by himself and Morley.* I cannot see any flaw either in the idea or in the execution of this experiment. But a possibility of escaping from the conclusion which it seemed to prove may be found in a brilliant suggestion made independently by Fitzerald † and by Lorentz,‡ of Leyden, to the effect that the motion of ether through matter may slightly alter its linear dimensions; according to which, if the stone slab constituting the sole plate of Michelson and Morley's apparatus has, in virtue of its motion through space occupied by ether, its lineal dimensions shortened one one-hundred-millionth § in the direction of motion, the result of the experiment would not disprove the free motion of ether through space occupied by the earth.

* *Phil. Mag.*, December 1887.

† Public Lectures in Trinity College, Dublin.

‡ *Versuch einer Theorie der electrischen und optischen Erscheinungen in bewegten Körpen.* Leiden, 1895.

§ This being the square of the ratio of the earth's velocity round the sun (30 kilometres per sec.) to the velocity of light (300,000 kilometres per sec.).

The Total Solar Eclipse of 28th May 1900.
By Thomas Heath, B.A.

(Read June 18, 1900.)

The Scottish Expedition to observe the Total Solar Eclipse of May 28 consisted of Professor Copeland and Mr J. B. M'Pherson, Engineer to the Royal Observatory, Edinburgh, who were sent out by the Joint Eclipse Committee of the Royal and Royal Astronomical Societies; Mr Franklin-Adams, who joined the party as a volunteer observer; and myself, who had the honour of being sent out by this Society. The special object which we had set before ourselves was, of course, the attempt, if possible, to add something, however little it might be, to the sum of known facts concerning the constitution of the solar corona. This problem, as I need not remind this Society, has occupied the minds of all students of solar physics for many years, and has formed the chief object of all eclipse expeditions since the middle of the century now drawing to a close, and I believe it is safe to say that every one of the short and fleeting opportunities of observing the corona with modern instruments and under modern conditions which have been afforded by the recurrence of total eclipses has been made the most of by an earnest band of observers since the famous eclipse of 1842 presented the problem as a burning question to the attention of astronomers. The introduction of the spectroscope and the possibilities which it presented of throwing new light on the subject still further increased the interest taken in the observation of the corona at the time of total eclipse, this being the only time at which such observation is possible.

It is not my intention now to enquire into what additional facts have been gleaned from the observation of successive eclipses, but if anyone were to enquire whether the great problem has yet been solved, it would be almost sufficient to point, in reply, to the ever-increasing number of observers who are attacking

the problem. The Indian Eclipse of 1898 must have held the record for the number of men and the extent of instrumental equipment taking part in the work; but I should think it more than probable, without presuming to say that I have made any estimate of numbers, that the eclipse of 1900 has beaten the record once more. The path of totality, crossing, as it did, both the New and Old worlds in regions easily accessible both to the traveller and to his heaviest baggage, rendered the various expeditions more like pleasant holiday tours than serious scientific undertakings. The whole line, from its commencement on the shores of the Pacific Ocean to its termination in Egypt, was more or less thickly studded with astronomical parties, armed with telescopes, spectroscopes, cameras, etc. The western part of the path of totality, where the line crosses from the Pacific coast of Mexico to the States of Louisiana and Virginia, was manned almost entirely by American astronomers, ever keen in the pursuit of science. So far as I am aware only one English party—that under the leadership of the Rev. Mr Bacon—ventured to cross the Atlantic to assist our American cousins. On the other hand, a large number of English expeditions stationed themselves on the line where it crossed the peninsula of Spain and Portugal. The Astronomer-Royal and assistants from Greenwich were at Ovar, some twenty miles south of Oporto, where the shadow track first enters European soil. The interior of Spain was occupied by at least three English parties—at Placencia, Noval Moval, and Manzanares—while the Scotch party found a resting-place at Santa Pola on the south-east coast, twelve miles south of the port of Alicante. At this station Sir Norman Lockyer also organised a camp, manned by three scientific assistants and a large body of officers and sailors belonging to H.M.S. *Theseus*. Inland from Santa Pola about ten miles, the old Moorish town of Elche was taken possession of for the time by a numerous contingent of French astronomers and one or two Englishmen. After passing Santa Pola the shadow crossed the Mediterranean Sea and struck land again at Algiers. Here quite a large number of astronomers were stationed, including many members of the British Astronomical Association: representing Oxford and Cambridge Universities were Professor Turner and Mr Newall; while

the Royal Astronomical Society was represented by Mr W. H. Wesley.

To turn to the special affairs of the Scotch party. The instrumental equipment consisted of the 40-foot telescope, which was under the special care of Dr Copeland, and was manipulated with great success by Mr M'Pherson. To me Dr Copeland kindly assigned the use of a new triple object-glass of 6-inch aperture, fitted with a whole plate camera and mounted on a heavy equatorial stand. Mr Franklin-Adams was armed with three portrait lenses, mounted in cameras to take plates of large size, with which he hoped to obtain photographs of the corona, showing the streamers to their utmost extent, and perhaps to find some trace of an intra-mercurial planet, if such has any existence. He was also provided with several ordinary cameras and a pair of long sensitive thermometers.

Dr Copeland's 40-foot telescope is well known to members of this Society, as it has already made its appearance at two previous eclipses, at one of which—the Indian Eclipse of 1898—a fine series of photographs of the corona and of spectra was secured.

The 6-inch Cooke triple object-glass is a newer instrument, and the desirability of trying the suitability of such an instrument for the production of photographs of the corona was the inducement which led to my joining the expedition to Santa Pola. The object-glass and camera were constructed by Messrs Cooke of York, and were fitted to a brass tube in our own workshop at Blackford Hill. The triple, or photo-visual, object-glass is made up of three lenses of Jena glass, combined in such a way as to bring the focus of the visual rays into practical coincidence with that of the photographic rays, so that the telescope can be used either for visual or photographic purposes without alteration. The combination is almost truly achromatic for all visual rays, the images of the moon's limb, or of such stars as Vega, showing no trace of the blue secondary spectrum so conspicuous in all other forms of so-called achromatic object-glasses. The instrument was completed only a few days before it was necessary to pack it up for transit to Spain. The interval, however, during which it was mounted at the Observatory was sufficient to allow of the position of the focus being determined with great care. Several trails of stars were

photographed, amongst them that of the double star ζ Ursæ Majoris, and the definition was found so good that the trail of the primary image was distinctly double all its length, though the components of the star differ in declination only by 12·6 seconds of arc, and the interval between the lines of the trails on the plate is only about $\frac{1}{150}$ of an inch. The focus was, of course, redetermined in Spain by means of trial photographs of the crescent moon, and its position was found to have remained unaltered.

Early in May the whole of the instruments were packed and forwarded for shipment on board the Orient Line Royal Mail Steamer *Oruba.* On May 11, three of the members of the party forgathered on the platform of St Pancras Station, bound for Tilbury Docks (Mr Franklin-Adams had preceded us by the P. and O. steamer). Here we met Sir Norman Lockyer and his party, who were, like ourselves, *en route* for Gibraltar by the *Oruba.* The journey out was distinctly uneventful. The wind was in our favour, and the Bay of Biscay was in such a gracious mood, that even unseasoned travellers like myself felt inclined to think that the discomfort popular report had prepared us for was a libel on the character of this smooth and smiling ocean. On our return journey it was again smooth, but I am assured by people who have crossed it more often than I have that it is not always in such a benign temper.

We reached Gibraltar on Wednesday, May 16, where we found H.M.S. *Theseus* awaiting our arrival. Mr Franklin-Adams joined us here and informed us that he had made all the necessary arrangements for our immediate transference on board the *Theseus.* This was accomplished with very little delay. Our heavy baggage having been placed in charge of Mr Daniells, one of the officers of the *Theseus,* we were thus relieved of all anxiety so far as it was concerned, and by noon of the same day the *Theseus* steamed out of harbour with Sir Norman Lockyer's party and ourselves comfortably settled on board. The voyage from Gibraltar to Santa Pola occupied just twenty-three hours, and was perhaps the most delightful part of our journey. The *Theseus* is a first-class cruiser, armed with twelve guns, and attached to the Mediterranean Squadron, and the kindness and attention paid us by Captain Tisdall and the officers soon made us feel quite at home. We

were shown all over the ship, had the working of the guns, from the two big 9·2-inch to the comparatively little Maxim-Nordenfeldt, ably explained to us, till we seemed to know all about them. The torpedo chambers, both above the water-line and below, where the great torpedo tubes lie ready at any moment to launch these dread engines of warfare at England's enemies, were specially interesting. The ship's engines and boilers, capable of working up to 10,000 horse-power, were explained to us in all their detail, from the great cylinders to the tiny speed indicator, a marvel of ingenuity in itself. But perhaps not the least interesting sight in this part of our journey was the view we had of the Sierra Nevada mountains, stretching along the south coast, still covered with snow and lit up by the bright southern sun. We cast anchor off Santa Pola the following day, Thursday, the 17th, in the forenoon, and here we experienced one of the few minor difficulties which fell to our lot. The big ship could not approach nearer the shore than about 1½ to 2 miles, and our landing was arranged to be carried out with the aid of the steam pinnace. In it we accordingly placed ourselves and our light baggage, including a certain leather bag containing two chronometers which had been entrusted to my special care. Under ordinary circumstances the steam pinnace as used by H.M. Navy is a most useful and seaworthy boat; on the present occasion we were all right till we approached the pier, and found ourselves in the thick of the surf caused by the stiff breeze which was blowing off sea. Fortunately our able coxswain at once grasped the situation, and seeing the impossibility of lying alongside the pier with any safety, he turned the boat's head to sea again and steamed out into comparatively smooth water. Here we awaited the arrival of a Spanish surf boat manned by two local fishermen, sent to us by the inhabitants of Santa Pola, who were waiting to welcome us on shore. In the end we were landed in safety, chronometers and all, with no worse experience than a slight shower-bath of salt water, which soon dried under the influence of the bright sunshine. As soon as possible after landing, we proceeded to look out for a site for our camp. Sir Norman Lockyer's camp had already been fixed upon by Mr Payn, a member of his party, who, travelling overland, had arrived a day or two earlier. It was situated on a flat piece of ground by the

sea-shore, and would have afforded ample room for our camp also, but as the soil was sandy, Dr Copeland considered it unsuitable for the heavier instruments we had to erect. We therefore fixed on a slight eminence overlooking the town, where we found a suitable field from which a crop of barley had recently been cut and was then being thrashed by the primitive process of treading out by mules and donkeys dragging stone-cogged rollers over a thrashing floor. The farmer-tenant of the field willingly placed it at our disposal, and we were fortunate enough to get possession also of an old disused and half tumbled-down stable in which we stored our instrument cases when they were sent ashore the morning after our arrival. The old stable also afforded us most grateful shelter from the hot sun in the middle of the day, and we even attempted to use it for a dark room for developing photographs at night. Owing, however, to the scarcity of water and the abundance of dust, as well as to the short time at our disposal after the eclipse for re-packing the instruments and sending them once more on board the *Theseus*, it was found impossible to develop any of the eclipse plates. The room, however, was found useful for developing a few less important photographs taken for focussing purposes. It is known in Santa Pola as "la casa del pleito," or the house of the lawsuit, on account of certain chancery proceedings, in which it has been for some years and is still involved. Our camp proved to be well suited for its temporary purpose. It commanded a good view of the western sky, and we found a rock foundation not far from the surface; it was only a short distance from the hotel we lived at, a matter of no small importance, as the adjustment of the instruments involved a good deal of night work; it was also in an elevated healthy situation, though at the same time well sheltered from winds likely to disturb the instruments.

On the 18th the instruments were landed, and the real work of laying out the ground, determining the meridian line, building the cement piers for the instruments to stand on, was commenced. This and the mounting of the instruments occupied us for several days, and by Friday, the 25th, everything may be said to have been ready, with the exception of the final adjustments. The week referred to, the 18th to the 25th, was thus a period of continuous work for every member of the expedition, broken, however, by

at least one incident, the receipt by telegram from Gibraltar of news of the relief of Mafeking. When we left home the strain of expectancy of this happy event was still dominating the public mind, and news of the relief was hourly looked for. The first enquiry on our arrival at Gibraltar was—What news of Mafeking? and the reply was—No news yet. Arrived at Santa Pola we were still in a state of some anxiety, till at last, on the 19th, we found the good news awaiting us on our appearance at the hotel after the morning's work. As British subjects sojourning on foreign soil, we found it impossible to restrain our feelings, and even thought it necessary to show our loyalty to the glorious empire to which we belong. The news was received with three hearty cheers, much to the amazement of our host, the people of the hotel, and the passing natives who happened to be loitering about the hotel door to look at the English astronomers. Whether they understood at the moment what it was all about, I know not, but they were not long in finding out that we were rejoicing over one more victory for British pluck. So far as I could understand, the sympathies of the Spaniards in the present Transvaal war are quite on the side of the Boers, and I presume there are reasons why Spanish human nature should entertain such feelings. During our stay at Santa Pola, however, this feeling was never allowed to show itself, and all through we were treated with the greatest courtesy and kindness, which manifested itself on more than one occasion in distinctly practical form.

I have now brought my narrative as far as May 25th. There were still two clear days, the 26th and 27th, before the day of the eclipse. These were occupied partly in drill, partly in putting final touches to the adjustments of the instruments, and generally in making final arrangements. On the 26th our camp, now completed, had the honour of being visited by the Civil Governor of the Province of Alicante, in which province Santa Pola is situated. A number of the French astronomers from Elche also visited us, and were received and entertained by Dr Copeland. The state of the weather naturally at this date engaged some of our attention, but I am bound to say, it never, at any time, caused us much concern. In the earlier days of our stay, there were on one or two occasions a few drops of

rain, and at least one night was cloudy. As the 28th approached the weather seemed, if anything, to improve, and culminated at the time of the eclipse in weather conditions which were everything that could possibly be desired. The brightness of the skies at night, indeed, formed a subject of comment amongst us all. The shorter duration of twilight than we are accustomed to at this season in this northern latitude enhanced the beauty of the evening sky. Evening after evening showed us the planet Venus, a strikingly beautiful object, then just at her position of greatest brilliancy. The brightness of the Milky Way also struck us all as very remarkable, especially a detached portion of it forming a little cloud not far from the constellation Scorpio. Scorpio itself seemed to remind us night after night how far South we had come from the scene of our regular work, for Antares, its chief star, only rises about 8° above the horizon at Edinburgh, whereas, at Santa Pola, it stared at us from the goodly elevation of over 25°.

The scene at our camp on the 28th was somewhat remarkable. We had fortunately enclosed the ground on which our instruments stood with a light wire fence, and, acting on the authority of the chief magistrate, or Alcalde of the town, had erected notices with the legend, "Se prohibe el paso." This we found quite sufficient to restrain the crowds of townspeople who daily assembled to watch our proceedings from encroaching on the space allotted to us. Every day, and all day long, the greatest interest had been taken in our work by crowds of people, who, I must say, conducted themselves in the most quiet and orderly fashion, and never in one single instance was the slightest attempt made to interfere with us in any way. It would have been cruel, however, if not impracticable on our part to attempt to restrain for ten long days the natural volubility of the Spanish tongue, and accordingly we heard enough of the language in the days preceding the eclipse to have made us all perfect masters of it, if we could only have taken reasonable advantage of the daily lessons we received. This was all very well before the eclipse, but it is evident that during the seventy-five seconds of totality, nothing would suit us better than that silence which is known to be golden. It was therefore arranged with the authorities,

and, I believe, published by the town crier, that at the call, "silencio," as totality was approaching, silence would be the best compliment our friends could pay us. The effect of this arrangement was most remarkable, and most creditable to the courteous character of the people. Before the eclipse and during the partial phase the volume of sound which reached our ears can be adequately compared only to the Tower of Babel, or the Falls of Niagara. But the moment one of our party, in stentorian tones, shouted the single word "silencio," the effect was like magic. Not a sound was heard from all the crowd of perhaps 2000 people till totality was passed, and we announced by our cheers that the great event was over and our programme successfully accomplished.

I would now like to say a word or two as to what the nature of our observations was, though, as my own negatives are still undeveloped, I am unable to say any more about them. Dr Copeland had arranged for a long series of exposures with the 40-foot telescope, and these were successfully made by Mr M'Pherson securing a series of ten short exposures on a sliding plate immediately before and after the beginning of totality, with the object, if possible, of obtaining the spectrum of the flash. Then three exposures on 18-inch plates of the corona, the prism being removed at the proper moment by an assistant. Next, another sliding plate received ten exposures with the prism as totality was about to end, and further five exposures on separate 18-inch plates of the spectrum of the returning crescent.

My own programme was less ambitious. All I attempted was four photographs of the corona during totality, with the 6-inch triple object-glass referred to before. The plates are whole-plate size, $8\frac{1}{2} \times 6\frac{1}{2}$ inches, and are of the triple-coated Sandell type on Chance's glass. I regret that I have not yet had time since my return home on Tuesday night to get these plates developed, but I hope to do so immediately, and to lay them before this Society at the earliest possible opportunity.

Mr Franklin-Adams' programme was made up of long exposure photographs of the corona with his three portrait lenses. Two of these were mounted on an equatorial stand belonging to the Royal Observatory, Edinburgh, and the third was mounted on the stand

of my telescope. What their success may have been I am unable to say, as they had not been developed when I last saw Mr Franklin-Adams. He also arranged for thermometric observations to be recorded by two of the midshipmen of the *Theseus*, who were regularly drilled at the camp in the details of their work. Mr Franklin-Adams had also several ordinary cameras of various apertures and focal lengths fixed on stands and adjusted to the sun's place at totality. These were manipulated for him by officers of the *Theseus*. The shadow bands were attended to by two of the junior officers. The end wall of our old stable, "la casa del pleito," dressed up in a new suit of white plaster, was made use of for this purpose, and though the conditions of the eclipse were not favourable for the purpose, the darkness never being great at any time, some success attended their efforts, four lines having been laid down in red and blue paint representing the direction of movement of the bands before and after totality.

I have now a few words to say as to the arrangements made for our reception by the astronomical and the civil authorities in Spain, and the assistance rendered to us by the officers and men of the *Theseus*. Before starting from home our plans were much facilitated by the kindness of the Director of the Madrid Observatory Señor Iñiguez, who supplied us with a series of beautiful maps showing the path of the eclipse, as well as that of 1905, over Spanish and Portuguese territory. This enabled us to determine with certainty the precise latitude and longitude of our chosen station, and allowed of the computation before starting of the exact times of the eclipse, its duration at Santa Pola, and the sun's azimuth at the moment of eclipse. This last was a matter of considerable importance, in view of the proper laying down of the 40-foot telescope. Señor Iñiguez also arranged with the Customs Authorities to admit our cases free of examination and without the annoyance of having to open them on landing, and also with the police authorities to give us every help possible. The assistance we received from the police was very great, though I must say that as guardians of the peace the necessity for their services was not very apparent. However, our camp was placed in charge of two members of the force known as the Guardia Civil, who, armed with Mauser rifles, and relieved at

suitable intervals, kept watch over us and our instruments, night and day. Their duties were not of a very onerous description, but they left us free from anxiety as to the safety of the instruments mounted in the open and protected only by waterproof sheets. Too much praise cannot be given to both the officers and men of the *Theseus* for the great services they rendered to us, first in assisting in the work of mounting the instruments and putting our camp into order, and secondly in the actual work of observing the eclipse. Every assistance we asked of them was given with the utmost enthusiasm and willingness. The navigating officer supplied us with a daily time signal, by dropping a ball on board the ship, giving us in this way a most satisfactory check on the going of our chronometers, of which we availed ourselves to the utmost extent. As I have already mentioned, two of the junior officers undertook charge of the shadow band observations. Two midshipmen read off the thermometers, and other officers exposed the numerous cameras under Mr Franklin-Adams' directions. The services of six or eight of the men we found invaluable. The "handy man" proved himself as capable of mounting equatorial telescope stands as he is of manipulating 4·7-inch guns. His cheerfulness and willingness to undertake any piece of work allotted to him was a constant source of pleasure to those of us who had to direct his energies.

By Thursday, the 31st, we had all our cases ready packed with the help of the sailors, and once more on board the *Theseus*. The same evening we bade farewell to our numerous Santa Pola friends, and before nightfall we were steaming down the east coast and leaving Santa Pola far behind us. Arrived at Gibraltar on Saturday morning, June 2, we found the Mediterranean fleet assembled there, and took up our place amongst them. Our homeward bound Orient Liner the *Cuzco* was not expected in Gibraltar till the 5th; we had therefore a few days to wait, which we employed in seeing something of the great fortress of Gibraltar. One day I and a companion spent at Ronda, an old Moorish town in the highlands of Malaga. The journey from Algeciras, on the bay opposite Gibraltar, took us by a new railway, to 2500 feet above sea-level, in about three hours. It is situated in a charming country, abounding in olives, which appears to be the principal

crop. There is much to see at Ronda in the shape of Moorish antiquities, and a fine bridge spans a gorge between cliffs some 300 feet high.

We left Gibraltar on the 6th, and reached Edinburgh on the 12th, feeling that we had no reason to be else than satisfied with our expedition.

In conclusion, my best thanks are due to the Lords of the Admiralty for permission to avail myself of all the advantages accorded to other observers, for transit for myself and instruments on board H.M.S. *Theseus*; to the captain and officers of the *Theseus* for their great kindness and assistance; to Señor Iñiguez, Director of the Madrid Observatory, for the use of maps and information sent prior to starting from home, and for his good offices in facilitating my business with the Customs and Police authorities; to Mr J. W. Cumming, H.M. Vice-Consul at Alicante, for much valuable aid; to Señor Francisco Bonmati y Mas, Alcalde of Santa Pola, and other local authorities for their thoughtful care on my behalf.

A Peculiar Set of Linear Equations. By Thomas Muir, LL.D.

(Read December 3, 1900.)

(1) It is easily seen that each of the equations of the set

$$\left.\begin{array}{rcrcrcrcl} x_1 & + & g_2x_2 & + & g_3x_3 & + & g_1 & = & 0 \\ g_1x_1 & + & x_2 & + & g_3x_3 & + & g_2 & = & 0 \\ g_1x_1 & + & g_2x_2 & + & x_3 & + & g_3 & = & 0 \end{array}\right\}$$

remains unaltered for each of the three interchanges

$$\begin{array}{ll} x_1 \leftrightarrows g_1 \,, & \qquad (1) \\ x_2 \leftrightarrows g_2 \,, & \qquad (2) \\ x_3 \leftrightarrows g_3 \,; & \qquad (3) \end{array}$$

and that the set as a whole is not altered by the simultaneous performance of the cyclical substitutions

(4)

If therefore we solve for x, in terms of g_1, g_2, g_3, and obtain

$$x_1 = \phi(g_1, g_2, g_3),$$

it must follow from (4) that

$$\begin{array}{rl} & x_2 = \phi(g_2, g_3, g_1), \\ \text{and} & x_3 = \phi(g_3, g_1, g_2). \end{array}$$

From this set of three, by the use of (1), we deduce

$$\left.\begin{array}{l} g_1 = \phi(x_1, g_2, g_3), \\ x_2 = \phi(g_2, g_3, x_1), \\ x_3 = \phi(g_3, x_1, g_2); \end{array}\right\}$$

from the same, by the use of (2), we deduce

$$\left.\begin{aligned} x_1 &= \phi(g_1, x_2, g_3), \\ g_2 &= \phi(x_2, g_3, g_1), \\ x_3 &= \phi(g_3, g_1, x_2); \end{aligned}\right\}$$

and from the same, by the use of (3), we deduce

$$\left.\begin{aligned} x_1 &= \phi(g_1, g_2, x_3), \\ x_2 &= \phi(g_2, x_3, g_1), \\ g_3 &= \phi(x_3, g_1, g_2); \end{aligned}\right\}$$

In the next place, by using simultaneously a pair of the three interchanges, the following three sets of results are obtained, viz. :—

$$\left.\begin{aligned} g_1 &= \phi(x_1, x_2, g_3), \\ g_2 &= \phi(x_2, g_3, x_1), \\ x_3 &= \phi(g_3, x_1, x_2); \end{aligned}\right\}.$$

$$\left.\begin{aligned} x_1 &= \phi(g_1, x_2, x_3), \\ g_2 &= \phi(x_2, x_3, g_1), \\ g_3 &= \phi(x_3, g_1, x_2); \end{aligned}\right\}$$

$$\left.\begin{aligned} g_1 &= \phi(x_1, g_2, x_3), \\ x_2 &= \phi(g_2, x_3, x_1), \\ g_3 &= \phi(x_3, x_1, g_2). \end{aligned}\right\}$$

Finally, by using all the three interchanges at the same time we obtain

$$\left.\begin{aligned} g_1 &= \phi(x_1, x_2, x_3), \\ g_2 &= \phi(x_2, x_3, x_1), \\ g_3 &= \phi(x_3, x_1, x_2). \end{aligned}\right\}$$

These eight sets of three equations may also be advantageously arranged as six sets of four, viz. :—

$$\begin{aligned} x_1 &= \phi(g_1, g_2, g_3) = \phi(g_1, x_2, g_3) = \phi(g_1, g_2, x_3) = \phi(g_1, x_2, x_3): \\ x_2 &= \phi(g_2, g_3, g_1) = \phi(g_2, x_3, g_1) = \phi(g_2, g_3, x_1) = \phi(g_2, x_3, x_1): \\ x_3 &= \phi(g_3, g_1, g_2) = \phi(g_3, x_1, g_2) = \phi(g_3, g_1, x_2) = \phi(g_3, x_1, x_2): \end{aligned}$$

$$\begin{aligned} g_1 &= \phi(x_1, x_2, x_3) = \phi(x_1, g_2, x_3) = \phi(x_1, x_2, g_3) = \phi(x_1, g_2, g_3): \\ g_2 &= \phi(x_2, x_3, x_1) = \phi(x_2, g_3, x_1) = \phi(x_2, x_3, g_1) = \phi(x_2, g_3, g_1): \\ g_3 &= \phi(x_3, x_1, x_2) = \phi(x_3, g_1, x_2) = \phi(x_3, x_1, g_2) = \phi(x_3, g_1, g_2). \end{aligned}$$

(2) The general set of n equations having this peculiarity is

$$\left.\begin{array}{rrrcr}
(g_1+x_1)+ & g_2x_2+ & g_3x_3+ & \ldots + & g_nx_n=0 \\
g_1x_1+ & (g_2+x_2)+ & g_3x_3+ & \ldots + & g_nx_n=0 \\
g_1x_1+ & g_2x_2+ & (g_3+x_3)+ & \ldots + & g_nx_n=0 \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
g_1x_1+ & g_2x_2+ & g_3x_3+ & \ldots & +(g_n+x_n)=0
\end{array}\right\},$$

and the number of derived sets of n equations in which the same function persistently appears is

$$1+C_{n,1}+C_{n,2}+\ \ldots\ +C_{n,n}$$
$$i.e. \qquad 2^n.$$

These $n\times 2^n$ equations may also be viewed as consisting of 2^{n-1} expressions for each of the $2n$ magnitudes $x_1, x_2, \ldots, x_n, g_1, g_2, \ldots, g_n$.

(3) When $n=3$ the persistent function, $\phi(a, \beta, \gamma)$, is

$$-\begin{vmatrix} a & \beta & \gamma \\ \beta & 1 & \gamma \\ \gamma & \beta & 1 \end{vmatrix} \div \begin{vmatrix} 1 & \beta & \gamma \\ a & 1 & \gamma \\ a & \beta & 1 \end{vmatrix};$$

and, generally,

$$\phi(a_1,a_2,a_3,\ldots,a_n)=-\begin{vmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ a_2 & 1 & a_3 & \ldots & a_n \\ a_3 & a_2 & 1 & \ldots & a_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n & a_2 & a_3 & \ldots & 1 \end{vmatrix} \div \begin{vmatrix} 1 & a_2 & a_3 & \ldots & a_n \\ a_1 & 1 & a_3 & \ldots & a_n \\ a_1 & a_2 & 1 & \ldots & a_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1 & a_2 & a_3 & \ldots & 1 \end{vmatrix}$$

A study of these two determinants, which are both functions of $a_1, a_2, \ldots, a_n$, and which may therefore be conveniently denoted by

$$N(a_1,a_2,\ldots,a_n) \qquad \text{and} \qquad D(a_1,a_2,\ldots,a_n)$$

is thus desirable.

(4) Taking D first we see that it may be defined as *a determinant in which all the elements of the principal diagonal are unity and in which no two non-diagonal elements situated in the same column*

are different; and the property of it which lies nearest the surface is that *it is a symmetrical function of all its variables.* In proof of this we have only to note that the transposition of the p^{th} and q^{th} rows, followed by the transposition of the p^{th} and q^{th} columns, has the effect of interchanging the two variables a_p and a_q and yet makes no alteration in the value of the determinant. This means, of course, that the order of the variables in $D(a_1,a_2,\ldots,a_n)$ is of no consequence.

(5) From this and the fact that, as the determinant form shows, the function D is linear in each of its variables, we should expect that D must be expressible in terms of the fundamental symmetric functions Σa_1, Σa_1a_2, $\Sigma a_1a_2a_3$, ... As a matter of fact it is found that

$$D = 1 - \Sigma a_1a_2 + 2\Sigma a_1a_2a_3 - 3\Sigma a_1a_2a_3a_4 + \ldots,$$

where it has to be noticed that the only missing member of the series is Σa_1. By way of proof of this second property we may proceed as follows, a special order, the 5th, being taken merely for the sake of brevity in writing:—

$$D(a_1,a_2,a_3,a_4,a_5) = \begin{vmatrix} 1 & a_2 & a_3 & a_4 & a_5 \\ a_1 & 1 & a_3 & a_4 & a_5 \\ a_1 & a_2 & 1 & a_4 & a_5 \\ a_1 & a_2 & a_3 & 1 & a_5 \\ a_1 & a_2 & a_3 & a_4 & \cdot \end{vmatrix} + \begin{vmatrix} 1 & a_2 & a_3 & a_4 \\ a_1 & 1 & a_3 & a_4 \\ a_1 & a_2 & 1 & a_4 \\ a_1 & a_2 & a_3 & 1 \end{vmatrix},$$

$$= a_5 \begin{vmatrix} 1 & a_2 & a_3 & a_4 & 1 \\ a_1 & 1 & a_3 & a_4 & 1 \\ a_1 & a_2 & 1 & a_4 & 1 \\ a_1 & a_2 & a_3 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 & \cdot \end{vmatrix} + D(a_1,a_2,a_3,a_4).$$

If the subsidiary determinant which here arises as the cofactor of a_5, and which therefore is the differential-quotient of $D(a_1,a_2,a_3,a_4,a_5)$ with respect to a_5, be expressed in terms of

the elements of the last row and their complementary minors, it is readily seen to be

$$-\alpha_1\begin{vmatrix} 1 & \alpha_3 & \alpha_4 & 1 \\ \alpha_2 & 1 & \alpha_4 & 1 \\ \alpha_2 & \alpha_3 & 1 & 1 \\ \alpha_2 & \alpha_3 & \alpha_4 & 1 \end{vmatrix} - \alpha_2\begin{vmatrix} 1 & \alpha_3 & \alpha_4 & 1 \\ \alpha_1 & 1 & \alpha_4 & 1 \\ \alpha_1 & \alpha_3 & 1 & 1 \\ \alpha_1 & \alpha_3 & \alpha_4 & 1 \end{vmatrix} - \alpha_3\begin{vmatrix} 1 & \alpha_2 & \alpha_4 & 1 \\ \alpha_1 & 1 & \alpha_4 & 1 \\ \alpha_1 & \alpha_2 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_4 & 1 \end{vmatrix} - \alpha_4\begin{vmatrix} 1 & \alpha_2 & \alpha_3 & 1 \\ \alpha_1 & 1 & \alpha_3 & 1 \\ \alpha_1 & \alpha_2 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & 1 \end{vmatrix},$$

where the cofactors of α_1, α_2, α_3, α_4 are like functions of α_2, α_3, α_4; α_1, α_3, α_4; α_1, α_2, α_4; α_1, α_2, α_3 respectively. Taking any one of them, say the first, we see that it is transformable into

$$\begin{vmatrix} 1-\alpha_2 & \alpha_3-1 & \cdot & \cdot \\ \cdot & 1-\alpha_3 & \alpha_4-1 & \cdot \\ \cdot & \cdot & 1-\alpha_4 & \cdot \\ \alpha_2 & \alpha_3 & \alpha_4 & 1 \end{vmatrix}$$

and therefore

$$= (1-\alpha_2)(1-\alpha_3)(1-\alpha_4). \qquad (\delta_1)$$

The subsidiary determinant above referred to is thus seen to be

$$\begin{aligned} &= -\alpha_1(1-\alpha_2)(1-\alpha_3)(1-\alpha_4) - \alpha_2(1-\alpha_3)(1-\alpha_4)(1-\alpha_1) \\ &\quad -\alpha_3(1-\alpha_4)(1-\alpha_1)(1-\alpha_2) - \alpha_4(1-\alpha_1)(1-\alpha_2)(1-\alpha_3), \\ &= -\Sigma_4\alpha_1 + 2\Sigma_4\alpha_1\alpha_2 - 3\Sigma_4\alpha_1\alpha_2\alpha_3 + 4\alpha_1\alpha_2\alpha_3\alpha_4 : \qquad (\delta_2) \end{aligned}$$

and consequently we have

$$\begin{aligned} D(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) &= \alpha_5\big(-\Sigma_4\alpha_1 + 2\Sigma_4\alpha_1\alpha_2 - 3\Sigma_4\alpha_1\alpha_2\alpha_3 + 4\alpha_1\alpha_2\alpha_3\alpha_4\big) \\ &\quad + D(\alpha_1,\alpha_2,\alpha_3,\alpha_4). \end{aligned}$$

If, therefore, the proposition hold good in regard to the case of the 4th order, that is to say, if

$$D(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = 1 - \Sigma_4\alpha_1\alpha_2 + 2\Sigma\alpha_1\alpha_2\alpha_3 - 3\alpha_1\alpha_2\alpha_3\alpha_4,$$

—and this is easily verified—we shall have

$$\begin{aligned} D(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) &= 1 - \big(\Sigma_4\alpha_1\alpha_2 + \alpha_5\Sigma_4\alpha_1\big) + 2\big(\Sigma_4\alpha_1\alpha_2\alpha_3 + \alpha_5\Sigma_4\alpha_1\alpha_2\big) \\ &\quad - 3\big(\alpha_1\alpha_2\alpha_3\alpha_4 + \alpha_5\Sigma_4\alpha_1\alpha_2\alpha_3\big) + 4\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5, \\ &= 1 - \Sigma_5\alpha_1\alpha_2 + 2\Sigma_5\alpha_1\alpha_2\alpha_3 - \ldots\ldots \qquad (\delta_3) \end{aligned}$$

which shows that it will hold also for the case of the 5th order. The proposition is thus established.

(6) The number of different kinds of terms in the final expansion of the determinant D of the n^{th} order is evidently

$$1 + C_{n,2} + C_{n,3} + C_{n,4} + \dots$$

which is equal to

$$(1+1)^n - C_{n,1} \quad i.e. \quad 2^n - n.$$

(7) By dividing in every case the p^{th} column by a_p there results

$$\frac{D(a_1,a_2,a_3,a_4,a_5)}{a_1a_2a_3a_4a_5} = \begin{vmatrix} \frac{1}{a_1} & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{a_2} & 1 & 1 & 1 \\ 1 & 1 & \frac{1}{a_3} & 1 & 1 \\ 1 & 1 & 1 & \frac{1}{a_4} & 1 \\ 1 & 1 & 1 & 1 & \frac{1}{a_5} \end{vmatrix},$$

$$= \begin{vmatrix} \frac{1}{a_1}-1 & 1-\frac{1}{a_2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{a_2}-1 & 1-\frac{1}{a_3} & \cdot & \cdot \\ \cdot & \cdot & \frac{1}{a_3}-1 & 1-\frac{1}{a_4} & \cdot \\ \cdot & \cdot & \cdot & \frac{1}{a_4}-1 & 1-\frac{1}{a_5} \\ 1 & 1 & 1 & 1 & \frac{1}{a_5} \end{vmatrix},$$

$$\begin{aligned} = \frac{1}{a_5} & \left(\frac{1}{a_4}-1\right)\left(\frac{1}{a_3}-1\right)\left(\frac{1}{a_2}-1\right)\left(\frac{1}{a_1}-1\right) \\ & + \left(\frac{1}{a_5}-1\right)\left(\frac{1}{a_3}-1\right)\left(\frac{1}{a_2}-1\right)\left(\frac{1}{a_1}-1\right) \\ & + \left(\frac{1}{a_5}-1\right)\left(\frac{1}{a_4}-1\right)\left(\frac{1}{a_2}-1\right)\left(\frac{1}{a_1}-1\right) \\ & + \left(\frac{1}{a_5}-1\right)\left(\frac{1}{a_4}-1\right)\left(\frac{1}{a_3}-1\right)\left(\frac{1}{a_1}-1\right) \\ & + \left(\frac{1}{a_5}-1\right)\left(\frac{1}{a_4}-1\right)\left(\frac{1}{a_3}-1\right)\left(\frac{1}{a_2}-1\right); \end{aligned}$$

$$\begin{aligned} \therefore D(a_1,a_2,a_3,a_4,a_5) = \sum_{r=1}^{r=4} & \frac{a_r}{1-a_r}(1-a_1)(1-a_2)(1-a_3)(1-a_4)(1-a_5) \\ & + (1-a_1)(1-a_2)(1-a_3)(1-a_4). \qquad (\delta_4) \end{aligned}$$

This when expanded contains a number of unnecessary terms, but it is useful as showing that when one of the variables is put

$=1$, the determinant resolves itself into binominal factors, which are got by subtracting each of the other variables from 1.

Writing ∂ for $\frac{1}{a_p}-1$, and subtracting and adding $\partial_1\partial_2\partial_3\partial_4$, we have the still more pleasing result

$$\frac{D(a_1,a_2,a_3,a_4,a_5)}{a_1a_2a_3a_4a_5} = \partial_1\partial_2\partial_3\partial_4\partial_5 + \Sigma\partial_1\partial_2\partial_3\partial_4 . \qquad (\delta_5)$$

(8) If we diminish each row of $D(a_1,a_2,a_3,a_4,a_5)/a_1a_2a_3a_4a_5$ by the row which follows it, and thereafter diminish each column by the column which follows it, the determinant resulting is an axisymmetric continuant, the identity being

$$\frac{D(a_1,a_2,a_3,a_4,a_5)}{a_1a_2a_3a_4a_5} = \begin{vmatrix} \partial_1+\partial_2 & \partial_2 & \cdot & \cdot & \cdot \\ \partial_2 & \partial_2+\partial_3 & \partial_3 & \cdot & \cdot \\ \cdot & \partial_3 & \partial_3+\partial_4 & \partial_4 & \cdot \\ \cdot & \cdot & \partial_4 & \partial_4+\partial_5 & \partial_5 \\ \cdot & \cdot & \cdot & \partial_5 & \partial_5+1 \end{vmatrix} . \qquad (\delta_6)$$

(9) Turning now to $N(a_1,a_2,a_3,a_4,a_5)$ we note first that it is obtainable from $D(a_1,a_2,a_3,a_4,a_5)$ by deleting the first column of the latter and substituting a_1, a_2, a_3, a_4, a_5. The first row and first column of N are thus identical, and a_1, instead of being as in D in every place except 1,1, occurs in that place only. This suggests the partition of $N(a_1,a_2,a_3,a_4,a_5)$ into the aggregate of terms containing a_1 and the aggregate of terms independent of a_1, the resulting identity being

$$N(a_1,a_2,a_3,a_4,a_5) = a_1D(a_2,a_3,a_4,a_5) + \begin{vmatrix} \cdot & a_2 & a_3 & a_4 & a_5 \\ a_2 & 1 & a_3 & a_4 & a_5 \\ a_3 & a_2 & 1 & a_4 & a_5 \\ a_4 & a_2 & a_3 & 1 & a_5 \\ a_5 & a_2 & a_3 & a_4 & 1 \end{vmatrix} .$$

Now the subsidiary determinant on the extreme right can, by the process of interchanging any two rows except the first, and subsequently interchanging the corresponding columns, be shown to be a symmetrical function of a_2, a_3, a_4, a_5,—say $f(a_2,a_3,a_4,a_5)$. It follows therefore that both the cofactor of a_1 in N and the

aggregate of terms independent of a_1 are symmetric functions of the remaining variables. This implies that the order in which a_2, a_3, a_4, a_5 are written in $N(a_1,a_2,a_3,a_4,a_5)$, is of no consequence.

(10) Expressing the subsidiary determinant, $f(a_2,a_3,a_4,a_5)$, of the preceding paragraph in terms of the elements of its first row and their complementary minors, we find that the latter have the same form as N, and that the determinant is equal to

$$- a_2N(a_2,a_3,a_4,a_5) - a_3N(a_3,a_4,a_5,a_2) - a_4N(a_4,a_5,a_2,a_3) - a_5N(a_5,a_2,a_3,a_4)\,.$$

There thus results

$$N(a_1,a_2,a_3,a_4,a_5) = a_1D(a_2,a_3,a_4,a_5) - \overset{\circ}{\sum} a_2N(a_2,a_3,a_4,a_5)\,. \qquad (\nu_1)$$

(11) Again, expanding the said subsidiary determinant in terms of binary products of the first-row elements and the first-column elements, we find it

$$= -a_2^2D(a_3,a_4,a_5) - a_3^2D(a_2,a_4,a_5) - \ldots$$

$$+ a_2a_3\begin{vmatrix} a_3 & a_4 & a_5 \\ a_3 & 1 & a_5 \\ a_3 & a_4 & 1 \end{vmatrix} + a_3a_2\begin{vmatrix} a_2 & a_4 & a_5 \\ a_2 & 1 & a_5 \\ a_2 & a_4 & 1 \end{vmatrix}$$

$$+ a_2a_4\begin{vmatrix} a_4 & a_3 & a_5 \\ a_4 & 1 & a_5 \\ a_4 & a_3 & 1 \end{vmatrix} + a_4a_2\begin{vmatrix} a_2 & a_3 & a_5 \\ a_2 & 1 & a_5 \\ a_2 & a_3 & 1 \end{vmatrix}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= - \Sigma a_2^2D(a_3,a_4,a_5) + \Sigma a_2a_3(a_2+a_3)(1-a_4)(1-a_5)\,.$$

Now it is easily shown that

$$-\Sigma a_2^2D(a_3, a_4, a_5) = -\Sigma a_2^2 + \Sigma a_2^2a_3a_4 - 2\Sigma a_2^2a_3a_4a_5\,,$$

and that

$$\Sigma a_2a_3(a_2+a_3)(1-a_4)(1-a_5) = \Sigma a_2^2a_3 - 2\,\Sigma a_2^2a_3a_4 + 3\Sigma a_2^2a_3a_4a_5,$$

It follows therefore by addition that the aggregate of terms independent of a_1 in $\mathrm{N}(a_1,a_2,a_3,a_4,a_5)$ is

$$-\Sigma a_2^2+\Sigma a_2^2a_3-\Sigma a_2^2a_3a_4+\Sigma a_2^2a_3a_4a_5\,, \qquad (\nu_2)$$

and that

$$\mathrm{N}(a_1,a_2,a_3,a_4,a_5) \;=\; a_1\mathrm{D}(a_2,a_3,a_4,a_5)-\Sigma a_2^2+\Sigma a_2^2a_3 \qquad (\nu_3)$$
$$-\Sigma a_2^2a_3a_4+\Sigma a_2^2a_3a_4a_5\,.$$

(12) The general theorem of which this is a case may be established by so-called 'mathematical induction.' Subtracting the first row of $f(a_2,a_3,a_4,a_5)$ from the last row, we have

$$f(a_2,a_3,a_4,a_5)=\begin{vmatrix} . & a_2 & a_3 & a_4 & a_5 \\ a_2 & 1 & a_3 & a_4 & a_5 \\ a_3 & a_2 & 1 & a_4 & a_5 \\ a_4 & a_2 & a_3 & 1 & a_5 \\ a_5 & . & . & . & 1-a_5 \end{vmatrix},$$

$$=(1-a_5)f(a_2,a_3,a_4)\;+\;a_5^2\begin{vmatrix} a_2 & a_3 & a_4 & 1 \\ 1 & a_3 & a_4 & 1 \\ a_2 & 1 & a_4 & 1 \\ a_2 & a_3 & 1 & 1 \end{vmatrix},$$

$$=(1-a_5)f(a_2,a_3,a_4)\;-\;a_5^2(1-a_2)(1-a_3)(1-a_4)\,.$$

If therefore the law in regard to $f(\quad)$ hold in the case of the third order, that is to say, if

$$f(a_2,a_3,a_4)=-\Sigma_3a_2^2+\Sigma_3a_2^2a_3-\Sigma_3a_2^2a_3a_4\,,$$

—and this is easily verified—we shall have

$$\begin{aligned} f(a_2,a_3,a_4,a_5) &= -\Sigma_3a_2^2+\Sigma_3a_2^2a_3-\Sigma_3a_2^2a_3a_4 \\ &\qquad +a_5\Sigma_3a_2^2-a_5\Sigma_3a_2^2a_3+a_5\Sigma_3a^2a_3a_4 \\ &\qquad -a_5^2+a_5^2\Sigma_3a_2-a_5^2\Sigma_3a_2a_3+a_5^2\Sigma_3a_2a_3a_4 \\ &= -\Sigma_4a_2^2+\Sigma_4a_2^2a_3-\Sigma_4a_2^2a_3a_4+\Sigma_4a_2^2a_3a_4a_5\,, \end{aligned}$$

which shows that it will hold also for the fourth order.

(13) The expansion of N in terms of simple symmetric functions having thus been obtained, the number of different kinds of terms

in the expansion is easily determined. In the case of the 4th order it is

$$(2^3-3) \;+\; (3+6+3) \qquad\qquad i.e.\ 17\,;$$

in the case of the 5th order it is

$$(2^4-4) \;+\; (4+12+12+4), \qquad\qquad i.e.\ 44\,;$$

and for the n^{th} order it is

$$\left(2^{n-1}-\overline{n-1}\right)+\left\{(n-1)+(n-1)(n-2)+(n-1)C_{n-2,2}+(n-1)C_{n-2\,3}+\ldots\right\}$$

$$\begin{aligned} \text{which } &= 2^{n-1}-(n-1)+(n-1)2^{n-2}, \\ &= (n+1)2^{n-2}-(n-1). \end{aligned}$$

(14) The D and any two of the N's associated with such a set of equations are connected by a simple relation, the only other magnitudes involved being the elements in the place 1,1 of the two N's. For, taking any two of the equations, say the second and third of a set of four, and subtracting, we have

$$(1-g_2)x_2-(1-g_3)x_3 \;+\; g_2-g_3 \;=\; 0\,;$$

and therefore by substituting for x_2 and x_3

$$(1-g_2)\mathrm{N}(g_2,g_3,g_4,g_1) \;-\; (1-g_3)\mathrm{N}(g_3,g_4,g_1,g_2) \;=\; (g_2-g_3)\mathrm{D}(g_1,g_2,g_3,g_4).$$

(15) Returning now to § 1 we see that the four expressions obtained for any one of the six quantities, $x_1, x_2, x_3, g_1, g_2, g_3$, give rise to six equations, four of which involve only four of the said quantities. Thus from the expressions for x_1 we have

$$\begin{aligned} \phi(g_1, g_2, g_3) &= \phi(g_1, x_2, g_3), \\ \phi(g_1, g_2, g_3) &= \phi(g_1, g_2, x_3), \\ \phi(g_1, x_2, g_3) &= \phi(g_1, x_2, x_3), \\ \phi(g_1, g_2, x_3) &= \phi(g_1, x_2, x_3), \end{aligned}$$

each of which involves only four quantities, while the others

$$\begin{aligned} \phi(g_1, g_2, g_3) &= \phi(g_1, x_2, x_3), \\ \phi(g_1, x_2, g_3) &= \phi(g_1, x_2, x_3), \end{aligned}$$

involve five each. Taking the first of these six, which involves g_1, g_2, g_3, x_2, and writing it in the form

$$\mathrm{N}(g_1, g_2, g_3)\cdot \mathrm{D}(g_1, x_2, g_3) \;-\; \mathrm{N}(g_1, x_2, g_3)\cdot \mathrm{D}(g_1, g_2, g_3) \;=\; 0,$$

we see that $g_2 - x_2$ must be a factor of the left-hand side. Now

$$\begin{aligned} N(g_1, g_2, g_3) &= (g_1 - g_3^2) \quad + (g_3^2 - g_3 g_1)g_2 + (g_3 - 1)g_2^2, \\ &= A + Bg_2 + Cg_2^2, \text{ say:} \end{aligned}$$

$$\begin{aligned} \text{and} \quad D(g_1, g_2, g_3) &= (1 - g_1 g_3) + (2g_1 g_3 - g_1 - g_3)g_2, \\ &= A' + B'g_2, \text{ say.} \end{aligned}$$

The above equation thus becomes

$$\begin{vmatrix} A + Bg_2 + Cg_2^2 & A' + B'g_2 \\ A + Bx_2 + Cx_2^2 & A' + B'x_2 \end{vmatrix} = 0,$$

which on the removal of $g_2 - x_2$ is easily reducible to

$$\begin{vmatrix} B + C(g_2 + x_2) & B' \\ A - Cg_2 x_2 & A' \end{vmatrix} = 0,$$

$$\text{or} \quad \begin{vmatrix} B & B' \\ A & A' \end{vmatrix} + C\begin{vmatrix} g_2 + x_2 & B' \\ -g_2 x_2 & A' \end{vmatrix} = 0.$$

From this the further factor C may be removed because

$$\begin{vmatrix} B & B' \\ A & A' \end{vmatrix} = \begin{vmatrix} B + A & B' + A' \\ A & A' \end{vmatrix} = C(g_1^2 g_3 + g_1 g_3^2 - g_1^2 - g_3^2);$$

consequently our equation takes the final form

$$(2g_1 g_3 - g_1 - g_3)g_2 x_2 + (1 - g_1 g_3)(g_2 + x_2) + (g_1^2 g_3 + g_1 g_3^2 - g_1^2 - g_3^2) = 0,$$

and is thus seen to be (1) symmetrical with respect to g_2 and x_2, (2) symmetrical with respect to g_1 and g_3, (3) linear in each of the two former, (4) a quadratic in the two latter. Solving for g_2 and x_2 we obtain, as we ought,

$$\begin{aligned} g_2 &= \phi(x_2, g_3, g_1), \\ x_2 &= \phi(g_2, g_3, g_1): \end{aligned}$$

so that we have the very interesting proposition :—

If $\phi(g_1, g_2, g_3) = \phi(g_1, x_2\ g_3)$ *and* g_2 *and* x_2 *be unequal, then* $g_2 = \phi(x_2, g_3, g_1)$ *and* $x_2 = \phi(g_2, g_3, g_1)$.

Arranging the equation as a quadratic in g_1 we may write it

$$(1 - g_3).g_1^2 + D'(g_3, g_2, x_2).g_1 + (g_3 g_2 x_2 - g_2 - x_2 + g_3^2) = 0,$$

and if the result of solution be

$$g_1 = \psi_1(g_3, g_2, x_2) \quad \text{or} \quad \psi_2(g_3, g_2, x_2)$$

where $\psi_1+\psi_2=\mathrm{D}'/(g_3-1)$, then by using the three other equations we have in all

$$\begin{aligned}
g_1 &= \psi_1(g_3, g_2, x_2) \quad \text{or} \quad \psi_2(g_3, g_2, x_2), \\
g_1 &= \psi_1(g_2, g_3, x_3) \quad \text{or} \quad \psi_2(g_2, g_3, x_3), \\
g_1 &= \psi_1(x_2, g_3, x_3) \quad \text{or} \quad \psi_2(x_2, g_3, x_3), \\
g_1 &= \psi_1(x_3, g_2, x_2) \quad \text{or} \quad \psi_2(x_3, g_2, x_2).
\end{aligned}$$

From these by cyclical substitution we shall obtain four similar expressions for g_2, g_3, and by the interchange of x's and g's four similar expressions for each of the three x_1, x_2, x_3. In regard to this interchange however it is important to note that the expressions obtained for any x are exactly those obtained for the corresponding g, the reason for which is apparent on looking at the above quadratic equation in g_1, where, on account of the original equations being symmetric with respect to g_1 and x_1, it is legitimate to substitute x_1 for g_1. If g_1 and x_1 be supposed different, the above twenty-four results may therefore be arranged as twelve pairs, viz. :—

$$\left\{\begin{aligned}
g_1 &= \psi_1(g_3, g_2, x_2) \quad \text{or} \quad \psi_2(g_3, g_2, x_2), \\
x_1 &= \psi_2(g_3, g_2, x_2) \quad \text{or} \quad \psi_1(g_3, g_2, x_2),
\end{aligned}\right.$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

From this there follow four expressions for each of the sums g_1+x_1, g_2+x_2, g_3+x_3, viz. :—

$$\begin{aligned}
g_1+x_1 &= \mathrm{D}'(g_3, g_2, x_2) \div (g_3-1), \\
&= \mathrm{D}'(g_2, g_3, x_3) \div (g_2-1), \\
&= \mathrm{D}'(x_2, g_3, x_3) \div (x_2-1), \\
&= \mathrm{D}'(x_3, g_2, x_2) \div (x_3-1).
\end{aligned}$$

(16) Writing the original set of equations in the form

$$\left.\begin{aligned}
x_1 + g_3x_3 + (g_2x_2+g_1) &= 0 \\
g_1x_1 + g_3x_3 + (x_2+g_2) &= 0 \\
g_1x_1 + x_3 + (g_2x_2+g_3) &= 0
\end{aligned}\right\}$$

and eliminating x_1 and x_3 we have

$$\begin{vmatrix} 1 & g_3 & g_2x_2+g_1 \\ g_1 & g_3 & x_2+g_2 \\ g_1 & 1 & g_2x_2+g_3 \end{vmatrix} = 0,$$

which must be the same equation as that of the preceding paragraph. The symmetry with regard to g_2 and x_2, and with regard to g_1 and g_3, is apparent; and the partition of the determinant into two gives immediately the value of x_2, and equally readily the value of g_2, if the element in the place 2,3 be written $g_2 + x_2$ instead of $x_2 + g_2$.*

* The peculiar set of equations dealt with in this short paper can scarcely have escaped notice until now. They were suggested to me while examining a problem set by Professor Nanson in the *Educational Times* for September 1900, viz., "*If*, $a=(x^2-y)/(1-xy)$, *and* $b=(y^2-x)/(1-xy)$, *prove that* $(a^2-b)/(1-ab)=x$ *and* $(b^2-a)/(1-ab)=y$."

Note on Dr Muir's Paper on a Peculiar Set of Linear Equations. By Charles Tweedie, M.A., B.Sc.

(Read December 17, 1900.)

§ 1. In Dr Muir's Paper on a Peculiar Set of Linear Equations (*communicated* December 3, 1900) there occur two Determinants of the n^{th} order, the expansions of which are given by Dr Muir. As the paper in question has so much to do with Symmetric Functions, the following simple method of obtaining their expansions may not prove uninteresting, based, as it is, upon the elementary theory of Symmetric Functions and the so-called Principle of Indeterminate Coefficients. The two Determinants given are :—

$$D = \begin{vmatrix} 1 & a_2 & a_3 & \cdots & a_n \\ a_1 & 1 & a_3 & \cdots & a_n \\ a_1 & a_2 & 1 & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & a_3 & \cdots & 1 \end{vmatrix}$$

and

$$N = \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & \cdots & a_n \\ a_2 & 1 & a_3 & \cdots & \cdots & a_n \\ a_3 & a_2 & 1 & \cdots & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & a_2 & a_3 & & & 1 \end{vmatrix}$$

§ 2. *Expansion of* D.—As Dr Muir points out, D is a symmetric function of $a_1, a_2, \ldots a_n$, for the interchange of a_p and a_q may be effected by interchanging first the p^{th} and q^{th} columns, and then the p^{th} and q^{th} rows, and the result of these operations on the determinant is to leave it unaltered in value. Moreover, the expansion must be linear in each of the a's. It must therefore be of the form,—

$$1 + A_1\Sigma a_1 + A_2\Sigma a_1a_2 + A_3\Sigma a_1a_2a_3 + \ldots .$$

To determine the coefficients, put $a_1 = a_2 = \ldots = a_n = a$. The expansion then becomes

$$1 + {}_nC_1A_1a + {}_nC_2A_2a^2 + \ldots \qquad + {}_nC_rA_ra^r + \ldots ,$$

while D is clearly $(1-a)^{n-1}(1+\overline{n-1}a)$.

The coefficient of a^r in the latter expression is

$$(-1)^r\left[{}_{n-1}C_r - (n-1)_{n-1}C_{r-1}\right]$$

$$i.e., \; -(-1)^r(r-1)_nC_r\,.$$

Hence

$$\begin{aligned} {}_nC_rA_r &= -(-1)^r(r-1)_nC_r\,, \\ i.e.,\ A_r &= -(-1)^r(r-1)\,. \end{aligned}$$

The expansion of the Determinant is therefore

$$1 - \Sigma a_1a_2 + 2\Sigma a_1a_2a_3 - 3\Sigma a_1a_2a_3a_4 + \;.\;.\;.\;.$$

§ 3. *Expansion of* N.—The coefficient of a_1 is $D(a_2, a_3, \ldots a_n)$, and the remaining terms form $N(0, a_2, a_3, \ldots a_n)$. Now this latter determinant is, when expanded, a symmetrical function of $a_2a_3 \ldots a_n$, for the interchange of a_p and a_q may be effected by the interchange of the p^{th} and q^{th} columns, followed by the interchange of the p^{th} and q^{th} rows (*vide* Dr Muir's paper). Let us note what *Types* of symmetric functions can occur, and let us select those that involve a_2. Now a_2 occurs only in the first and second columns. If the type contains a_2^2 it must be linear in other variables, and if it contain a_2 and, say, a_4 as from the first and second columns, then it must contain a_4 again, since by taking a_4 from the first column we are prevented from taking the constituent 1 from the fourth column. This term is also linear in any other variables. Finally, there is no term independent of the variables.

The expansion of $N(0a_2a_3 \ldots a_n)$ must therefore be of the form

$$A_2\Sigma a_2^2 + A_3\Sigma a_2^2a_3 + A_4\Sigma a_2^2a_3a_4 + \;.\;.\;.\;.\;.$$

To determine the coefficients, put $a_2 = a_3 = \ldots = a_n = a$. The determinant then becomes

$$\begin{vmatrix} 0 & a & a & \cdots & a \\ a & 1 & a & \cdots & a \\ a & a & 1 & \cdots & a \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a & a & a & \cdots & 1 \end{vmatrix}_n = \begin{vmatrix} 1 & a & a & \cdots & a \\ a & 1 & a & \cdots & a \\ a & a & 1 & \cdots & a \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a & a & a & \cdots & 1 \end{vmatrix}_n - \begin{vmatrix} 1 & a & a & \cdots & a \\ a & 1 & a & \cdots & a \\ a & a & 1 & \cdots & a \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a & a & a & \cdots & 1 \end{vmatrix}_{n-1}.$$

in which, by § 2, the coefficient of a^r is

$$-(-1)^r\left[r_nC_r - r_{n-1}C_r\right] = -(-1)^r r_{n-1}C_{r-1}\,.$$

But $\Sigma a_2^2 a_3 \ldots a_r$ becomes at the same time $r_{n-1}C_{r-1}a^r$.

Hence

$$A_r = -(-1)^r,$$

and the expansion in question is

$$-\Sigma a_2^2 + \Sigma a_2^2 a_3 - \Sigma a_2^2 a_3 a_4 + \ldots .$$

Hence, finally,

$$N = a_1 \times D(a_2 a_3 \ldots a_n) - \Sigma a^2 + \Sigma a_2^2 a_3 - \ldots .$$

Note on Pairs of Consecutive Integers the Sum of whose Squares is an Integral Square. By Thomas Muir, LL.D.

(Read January 21, 1901.)

(1) The solution of the problem of finding such pairs of integers is not a thing of yesterday, as may be seen by consulting Hutton's translation of Ozanam's *Recreations*, i. pp. 46–8 (1814). It may be enunciated thus:—

If p_r/q_r *be the* r^{th} *convergent to* $\sqrt{2}$, *then* $p_r p_{r+1}$ *and* $2q_r q_{r+1}$ *are consecutive integers, and the sum of their squares is equal to* $(q^2_r + q^2_{r+1})^2$ *or* $(q_{2r+2})^2$. . . . . (1)

(2) By introducing the idea of a continuant,—which enables us to leave out any direct reference to $\sqrt{2}$,—we have the alternative form of enunciation:—

If the continuant $(2, 2, 2, \ldots)$ *be denoted by* a, b, c *when the number of* 2's *is* $r-1$, r, $r+1$ *respectively, then* $(a+b)(b+c)$ *and* $2bc$ *are consecutive integers, and the sum of their squares is equal to the square of* b^2+c^2. . . . . (2)

(3) Neither of these enunciations, however, indicates which of the two consecutive integers is the less;* and the merit of Mr Christie's enunciation (*Math. Gazette*, i. p. 394) arises from the fact that he has hit upon a general expression for the less of the two. The most striking way of putting his result is as follows:—

The solution of the equation $x^2+(x+1)^2=y^2$ *in integers is*

$$\left.\begin{aligned} x &= 2_0+2_1+2_2+\ \ldots\ +2_{2r-1} \\ y &= 2_{2r} \end{aligned}\right\},$$

where 2_r *is the simple continuant of the* r^{th} *order whose diagonal elements are all* 2's. . . . . (3)

(4) By way of proof of (2) we note that two properties of continuants give $ac-b^2=\pm 1$ and $c=2b+a$; that consequently

* The first is less when r is even, and the second when r is odd.

$$\begin{aligned}(a+b)(b+c)-2bc &= ab+b^2+ac-b(2b+a),\\ &= ac-b^2,\\ &= \pm 1;\end{aligned}$$

and that the well-known identity

$$\{a^2+4ab+3b^2\}^2+\{2ab+4b^2\}^2 = \{a^2 + 4ab+5b^2\}^2$$

gives

$$\{(a+b)(b+c)\}^2+\{2bc\}^2 = (b^2 + c^2)^2.$$

(5) As $\sqrt{2}=1+\frac{1}{2+}\frac{1}{2+}\ldots$, it follows that q_r, q_{r+1} are the same as b, c; and as a law of continuants gives

$$(1, 2, 2, 2, \ldots) = (2, 2, 2, \ldots) + (2, 2, \ldots)$$

we have

$$\begin{aligned}p_r &= q_r + q_{r-1} = b+a,\\ \text{and}\quad p_{r+1} &= q_{r+1}+q_r = c+b.\end{aligned}$$

The identity of (1) and (2) is thus apparent.

(6) The curious proposition which forms the basis of Mr Christie's improvement is to the effect that

$$\begin{aligned}&2_0+2_1+2_2+\ldots+2_{2r-1}\\ &=2\cdot 2_{r-1}\cdot 2_r \quad\text{or}\quad (2_{r-1}+2_r)(2_{r-2}+2_{r-1})-1 \quad\text{when } r \text{ is even},\\ &=2\cdot 2_{r-1}\cdot 2_r-1 \quad\text{or}\quad (2_{r-1}+2_r)(2_{r-2}+2_{r-1}) \quad\text{when } r \text{ is odd}.\end{aligned}$$

For the purposes of proof suppose the proposition to hold for $r=2s$, —that is, suppose

$$2_0+2_1+2_2+\ldots+2_{4s-1}=2\cdot 2_{2s-1}\cdot 2_{2s}.$$

From this we have of course

$$\begin{aligned}2_0+2_1+2_2+\ldots+2_{4s-1} &= 2\cdot 2_{2s-1}\cdot 2_{2s} + 2_{4s} + 2_{4s+1},\\ &= 2\cdot 2_{2s-1}\cdot 2_{2s} + (2_{2s}^2 + 2_{2s+1}^2) + 2_{2s}(2_{2s+1}+2_{2s-1}),\\ &= 2_{2s-1}\{2\cdot 2_{2s}+2_{2s-1}\} + 2_{2s}^2 + 2_{2s}(2_{2s+1}+2_{2s-1}),\\ &= 2_{2s-1}\cdot 2_{2s+1} + 2_{2s}\cdot 2_{2s+1} + 2_{2s}^2 + 2_{2s}\cdot 2_{2s-1},\\ &= (2_{2s+1}+2_{2s})(2_{2s-1}+2_{2s});\end{aligned}$$

and this we know otherwise (§ 3, footnote)

$$=2\cdot 2_{2s+1}\cdot 2_{2s} - 1.$$

Similarly, by adding $2_{4s+2}+2_{4s+3}$, we shall obtain

$$2_0+2_1+2_2+\ldots+2_{4s+3}=(2_{2s+2}+2_{2s+1})(2_{2s+1}+2_{2s})-1,$$

which we know otherwise (§ 3, footnote) equals $2\cdot 2_{2s+2}\cdot 2_{2s+1}$. It is thus clear that if the proposition hold for any particular case where r is even, it must hold for the next two cases, and therefore for the next two, and so on; and as its validity for the case $r=2$ is readily verified, the proposition may be considered to be established.

(7) When we have got one instance of an integer whose square, together with the square of the next higher integer, gives an integral square, there is a very simple means of getting the next instance. The theorem is:—

If a *be an integer such that* $a^2+(a+1)^2=z^2$, *where* z *is integral, then* $3a+1+2z$ *is the next integer of this kind.*

To establish this we have to show that

$$3(2_0+2_1+2_2+\ldots+2_{2r-1})+1+2\cdot 2_{2r}=2_0+2_1+2_2+\ldots+2_{2r+1},$$

that is, that

$$2(2_0+2_1+2_2+\ldots+2_{2r-1})+1+2_{2r}=2_{2r+1}.$$

Now, if we know one case of this to be true, we can immediately prove the next case; for, suppose that

$$2(2_0+2_1+2_2+\ldots+2_{2m-1})+1+2_{2m}=2_{2m+1};$$

then by adding $2_{2m}+2_{2m+1}$ we obtain

$$2(2_0+2_1+2_2+\ldots+2_{2m})+1+2_{2m+1}=2\cdot 2_{2m+1}+2_m=2_{2m+2}.$$

It remains only to show that it is true when $m=1$, and this is self-evident.

(8) From the foregoing we have

$$\begin{aligned}2_0+2_1+2_2+\ldots+2_{2r-1}&=\tfrac{1}{2}(2_{2r+1}-2_{2r}-1),\\&=\tfrac{1}{2}(2_{2r}+2_{2r-1}-1);\end{aligned}$$

and we are thus led to the theorem:—

The solution of the equation $x^2+(x+1)^2=y^2$ *in integers is*

$x = \frac{1}{2}(2_{2r} + 2_{2r-1} - 1)$, $y = 2_{2r}$, *where* 2_r *stands for the continuant* $(2, 2, 2, \ldots)$ *of the* r^{th} *order.*

Apart from all that precedes this can be proved in a line or two. For, by substitution,

$$\begin{aligned} x^2 + (x+1)^2 &= \tfrac{1}{4}(2_{2r} + 2_{2r-1} - 1)^2 + \tfrac{1}{4}(2_{2r} + 2_{2r-1} + 1)^2, \\ &= \tfrac{1}{2}2_{2r}^2 + \tfrac{1}{2}2_{2r-1}^2 + \tfrac{1}{2} + 2_{2r}2_{2r-1}, \\ &= \tfrac{1}{2}2_{2r}^2 + \tfrac{1}{2}2_{r-1}(2_{2r-1} + 2\cdot 2_{2r}) + \tfrac{1}{2}, \\ &= \tfrac{1}{2}2_{2r}^2 + \tfrac{1}{2}(2_{2r-1}\cdot 2_{2r+1} + 1), \\ &= \tfrac{1}{2}2_{2r}^2 + \tfrac{1}{2}2_{2r}^2 = 2_{2r}^2, \\ &= y^2. \end{aligned}$$

It is scarcely possible to think of the whole matter being put more simply or in shorter compass than this.

The Seaweed *Ulva latissima*, and its relation to the Pollution of Sea Water by Sewage. By Professor Letts and John Hawthorne, B.A., Queen's College, Belfast. (With Three Plates.)

(Read March 4, 1901.)

For a number of years a very serious nuisance has arisen from the 'sloblands' of the upper reaches of Belfast Lough during the summer and early autumn—the stench at low tide being often quite overpowering, and the air heavily charged with sulphuretted hydrogen.

A precisely similar nuisance, though not of the same magnitude, arises from the sloblands in the northern portion of Dublin harbour.

This nuisance, in Belfast at all events, has been supposed by many people to be due to sewage matters actually deposited on the slobland, but it requires but slight observation to prove that this can scarcely be the true explanation, for without doubt the nuisance is intimately associated with deposits of green seaweed, consisting almost entirely of the *Ulva latissima*, or, as it is commonly called, the 'Sea Lettuce.' *

* That others have noticed the occurrence of this seaweed in polluted sea water, and the nuisance which may arise from it, is shown by the following letter which we received from Professor Hartley, F.R.S., of the Royal College of Science, Dublin, during our investigation on the subject:—

"Professor Johnson has shown me your letter *in re* the sewage of Belfast and the shore weed. *That weed is never seen on any shore unless sewage runs into the water.* The stronger the sewage and the greater its volume, the more luxuriant its growth. I have observed this during the last twenty years in England, Wales, Ireland, and Scotland. About eight years ago I washed some of the weed in fresh sea water and placed it in a bottle of the same. In about twenty-four hours the bottle was opened and the contents found to be in an exceedingly offensive state.

"A paper of mine in the *Proc. Roy. Soc. Edinburgh*, session 1895-96, touches upon this matter."

Nothing, however, appears to have been published on the subject, and we are under the impression that most botanists consider *Ulva latissima* as characteristic rather of *brackish* than of polluted sea water.

In the upper reaches of Belfast Lough this weed grows in abundance, and during high winds or gales it is washed ashore, often in enormous quantities, forming banks which are frequently two or three feet thick, and extend at times for miles along the coast, especially on the southern shore.

Once deposited, these layers of weed often remain more or less stationary in the shallow bays or pools of the neighbourhood for months, and under these circumstances, and particularly in warm weather, rapid putrefaction occurs, and a perfectly intolerable stench arises, which is perceptible over a wide area, and seriously affects, not only the comfort of the inhabitants of the district, but the value of their property also.

. . . .

The investigation, the results of which we describe in the following pages, was originally undertaken with the view merely of ascertaining the cause of the nuisance arising from the slob-lands of Belfast. But we were gradually led into a more extended inquiry, which has embraced not only a study of the chemical changes which occur when *Ulva latissima* ferments, but in addition, an examination of the composition and characters of the weed itself, the isolation of the products of its fermentation, and attempts to isolate the particular organisms giving rise to these products; and finally we have endeavoured to ascertain, both experimentally and by an examination of localities in which the weed is either present in quantity or is virtually absent, the relationship of *Ulva latissima* to the pollution of sea water by sewage.

For the sake of clearness and convenience we shall give the results of our inquiry into these different questions in a somewhat different order from that in which they were obtained.

The Chemical Changes which occur when Ulva latissima *ferments.*

A quantity of the fresh weed was carefully washed in several changes of ordinary tap water until free from shells and debris of various kinds,* and it was then distributed between two flasks, one of which was filled with tap water and the other with sea

* The weed, as obtained by us from the Belfast foreshore, was nearly always infested with minute spiral shell-fish, which feed upon it and eat out circular holes.

water, care being taken to get rid of all air adhering to the weed. A well fitting (paraffined) cork was then attached to each of the flasks, and through the cork a gas delivery tube passed, which dipped into a small mercury pneumatic trough and under an inverted test-tube full of mercury. The flasks, with their attachments, were then left in the laboratory at ordinary (winter) temperatures.

After some six weeks the contents of the flask containing the *ulva* and sea water began to evolve gas, and a few days later they blackened, while those of the flask containing *ulva* and tap water gave off gas some days later, and no blackening subsequently occurred.

Some of the liquid from the first flask was driven over along with the gas, and when the test-tube became full of the latter, the liquid escaped on to the surface of the mercury in the pneumatic trough. It was found to be strongly acid, and as it evaporated, smelt of butyric acid. The gases from this flask were examined after an interval of about three months had elapsed since starting the experiment, and were found to consist mainly of hydrogen, carbonic anhydride, sulphuretted hydrogen, and nitrogen.

These preliminary experiments gave a distinct clue to the nature of the chemical changes which the weed suffers when it rots on the foreshore in a moist condition, as well as to the cause of the nuisance to which it then gives rise.

It is clear that an acid is produced in the first stage of the fermentation process, while at a later period, and probably by a distinct fermentative act, sulphides and sulphuretted hydrogen are formed, either by the reduction of the sulphates present in the weed itself or in the sea water, or from the albuminoids contained in the former,—these sulphides reacting on the iron compounds in the tissues of the weed to give ferrous sulphide. The latter would no doubt be attacked by the acid, with evolution of sulphuretted hydrogen, and thence the nuisance. As a result of these preliminary experiments, we decided to investigate the quantitative composition of the gases evolved from the fermenting *ulva*, and also to isolate and identify the butyric acid.

To obtain the gases, the same arrangement was employed as before, only the flasks were placed as soon as charged in an incu-

bator at blood heat. Under these circumstances gases began to come off in 48 hours, and were then rapidly evolved.

The following analyses were made :—

Analysis of Gases evolved from Ulva latissima *fermenting in Sea Water.*

	I. (Collected 5 days after incubation, at 37° C.)	II. (Collected 12 days after incubation, at 37° C.)
Volume of gas taken, . . .	14·0 c.c.	16·55
After addition of potash, . .	8·0 ,,	8·45
,, pyro, . . .	8·0 ,,	8·45
Oxygen then added, . . .	13·7 ,,	11·55
After explosion, ...	10·0 ,,*	7·65*
CO_2 found, ...	6·0 c.c.	8·10
O_2 ,, ...	none	none
H_2 ,, ...	7·8 c.c.	8·23
N_2 ,, ...	0·2 ,,	0·22
	14·0 c.c.	16·55
Percentage composition.		
CO_2	42·8 c.c.	48·94
H_2	55·7 ,,	49·73
N_2	1·5 ,,	1·33
	100·0 c.c.	100·00

No sulphuretted hydrogen was present in the gases, which, as their analyses indicate, consisted entirely of carbonic anhydride and hydrogen. Owing to the solubility of carbonic anhydride in water, it was to be expected that the gases collected at first would contain a lower proportion of this constituent than was actually evolved. Only when the liquid in the flask had become saturated with carbonic anhydride would the gaseous products of the fermentation make their way into the collecting tube in their proper proportions, and this state appears to have been reached when the second analysis was made. Its results show the carbonic anhydride and hydrogen to be present practically in the same proportions by volume or in equi-molecular proportions.

* The gas remaining after explosion contained no CO_2.

The fermentation of grape sugar by the *Bacillus butyricus* is usually represented by the equation

$$C_6H_{12}O_6 = 2CO_2 + 2H_2 + C_4H_8O_2;$$

and, as we have already mentioned, before we had made any gas analyses, the production of butyric acid had been indicated.

A qualitative examination of the contents of the flask after fermentation gave further evidence of the production of the acid, for on distilling them with sulphuric acid an acid liquid passed over, and this, when neutralised with soda and evaporated to dryness, gave a solid residue, which, when warmed with strong sulphuric acid, emitted a distinct odour of butyric acid. Also when it was warmed with strong sulphuric acid and alcohol, the characteristic odour of butyric ether became apparent.

We should probably not have pursued the question further had it not been for the results of a quantitative analysis of what we supposed to be calcium butyrate obtained as follows:—

Some of the *ulva* was well washed and packed into a flask, which was then filled with sea water and the mixture fermented in an incubator at 37° C. until gas evolution ceased. The liquid was then strained off from the seaweed through a cloth filter, distilled with sulphuric acid, the distillate boiled with excess of calcium carbonate, filtered, and evaporated to dryness. Weighed portions of the carefully dried residue were then ignited with strong sulphuric acid, with the following results:—

0·2546 grm. gave	0·1883 grm. $CaSO_4$	= 0·0554 grm. Ca	= 21·75 % Ca.
0·3695 ,,	0·2681 ,,	= 0·0788 ,,	= 21·34 ,,

Anhydrous calcium acetate requires	25·32 %	Ca.
,, ,, propionate requires	21·50	,,
,, ,, butyrate ,,	18·70	,,

These results indicated that propionic and not butyric acid had been produced, and the matter seemed worth further investigation, as there appears to be some doubt as to a propionic fermentation from crude vegetable substances, and it is certainly not mentioned in modern chemical or bacteriological text-books. On the other hand, in the older chemical works such a fermentation is mentioned. Thus in

Wurtz's *Dictionnaire de Chimie* the following statements are made under the article *Acide Propionique*:—

(1) Redtenbacher obtained propionic acid by exposing glycerine and yeast for several months at 20–30° C. [*Liebig's Annalen*, 57 (1845), p. 174.]

(2) Keller, by fermenting bran and scraps of leather with chalk. [*Liebig's Annalen*, 73 (1850), p. 205.]

(3) Putrefaction of peas or lentils gives propionic and butyric acids. [Boehme, *Journ. prakt. Chem.*, 40 (1847), p. 278.]

(4) Fermentation of calcium tartrate. [Noellner, *Liebig's Annalen*, 38 ( ), p. 299. Nicklès, *ibid.*, 61 ( ), p. 343. Dumas, Malaguti, and Leblanc, *Comp. Rend.*, 25 ( ), p. 781.]

(5) Propionic acid is stated to be produced by the fermentation of glycerine and of sugar under certain circumstances. [Strecker's *Lehrbuch der organischen Chemie*, 5th edition (1867), p. 159.]

While in Richter's *Organic Chemistry* (English translation, 1900) none of these methods are mentioned, the only process of a similar kind for the production of the acid there alluded to being the fermentation of calcium malate and lactate.

We therefore decided to prepare a quantity of the acid or acids which the fermenting *ulva* gives rise to, and to submit them to a careful examination.

Accordingly, a considerable quantity of the well-washed seaweed was fermented with sea water at 37° C. in large flasks until no further evolution of gases occurred, which required about fourteen days. The resulting fluid was drained off from what remained of the seaweed and distilled with dilute sulphuric acid until traces of hydrochloric acid began to come over. The distillate was neutralised with caustic potash and evaporated to dryness, when about 25 grms. of solid residue were obtained. Experiments with weighed quantities of a known sample of potassium butyrate indicated that the best method for extracting the acid was to treat a strong aqueous solution of the salt with sulphuric acid, and then to extract with ether; distillation of the dry salt with concentrated sulphuric acid leading to considerable charring and loss.

The dried residue was therefore dissolved in 80 c.c. of water, the

resulting solution filtered and mixed in a separating funnel with 40 c.c. of strong sulphuric acid, when an oily liquid rose to the surface. The contents of the separating funnel were then extracted six times with well-washed ether, the ethereal extracts filtered and distilled from a water-bath.

The remaining liquid was submitted to fractional distillation, and after three fractionations, the bulk distilled over between 140–150° C. The lower boiling portions were treated with phosphoric anhydride and separately fractionated.

They yielded three fractions, which were collected at the following temperatures:—

(1) 110°–125°
(2) 125°–150°
(3) 150°–165°

The main portion of the distillate from the first fractionation weighed 4·7 grms., and had an odour which closely resembled that of a known sample of propionic acid.

It was boiled with water and barium carbonate until neutralised, the resulting solution filtered and evaporated on a water-bath to a syrup. Its behaviour now was curious. Some of the syrup was dissolved in a little alcohol and ether was then added, when it was reprecipitated apparently in the same condition. A drop of the syrup exposed for several hours also dried up to a gummy mass which refused to crystallise. But the main quantity of syrupy liquid suddenly solidified on stirring. The resulting crystalline mass was washed with cold alcohol two or three times and was then dried. It behaved in precisely the same way as a known specimen of barium propionate. Thus it readily dissolved in cold water; and on adding alcohol to a concentrated solution thus obtained, brilliant crystals separated out, which, when examined under the microscope, had very characteristic forms, being either quadratic octohedra or combinations of the octohedra with quadratic prisms. Its analysis, however, showed that it contained small quantities of an impurity which obstinately adhered to it, as the following figures show:—

Analysis of Barium Salt dried at 100° C. *until of constant weight.*

(1) 0·5153 grm. gave 0·4190 grm. $BaSO_4$ = 0·2464 grm. Ba = 47·8% Ba.

[The salt had been washed several times with cold alcohol.]

(2) 0·2000 grm. gave 0·1623 grm. $BaSO_4$ = 0·09543 grm. Ba = 47·71%.

[The salt, after washing with cold alcohol, had been boiled with alcohol.]

(3) 0·2648 grm. gave 0·2156 grm. $BaSO_4$ = 0·1268 grm. Ba = 47·87% Ba.

[In addition to the treatment to which (2) had been submitted, this portion of the salt had been recrystallised from water by the addition of alcohol.]

(4) 0·3216 grm. gave 0·2622 grm. $BaSO_4$ = 0·1542 grm. Ba = 47·93% Ba.

[This salt was obtained from the mother liquors of (3), but was recrystallised from water and alcohol.]

Obtained :—		Calculated for :—	
(1)	47·8	$Ba(C_2H_3O_2)_2$	53·72
(2)	47·71	$Ba(C_3H_5O_2)_2$	48·41
(3)	47·87		
(4)	47·93	$Ba(C_4H_7O_2)_2$	44·05

In order still further to identify the propionic acid, a quantity of the silver salt was obtained by decomposing a solution of 2 grams of the barium salt with the equivalent quantity of silver nitrate. The resulting white precipitate was washed on a filter until the washings gave no precipitate with sulphuric acid, and crystallised from hot water.

Analysis of Silver Salt dried in the desiccator.

(1) 0·232 grm. gave on ignition 0·1305 grm. Ag = 56·25%.

(2) 0·0720 grm. " " 0·0400 " = 55·55%

[obtained from the mother liquors of No. 1].

(3) 0·3633 grm. gave on ignition 0·2141 grm. Ag = 58·98%

[obtained by further evaporation of the mother liquors from No. 2].

% Ag obtained :—		% Ag calculated for :—	
(1)	56·25	$AgC_2H_3O_2$	64·66
(2)	55·55	$AgC_3H_5O_2$	59·66
(3)	58·93	$AgC_4H_7O_2$	55·38

It is evident from these figures that some butyrate clings obstinately to the propionate, and owing to its relative insolubility is precipitated first, the bulk of the propionate being found in the mother liquors.

A qualitative reaction was next employed for the identification of the propionic acid.

If this acid is boiled with excess of litharge and the solution allowed to remain in the cold for some time in contact with the litharge, a basic lead salt is produced, which is more soluble in cold than in hot water, and hence is precipitated on boiling the solution.

Comparative experiments tried both with a known sample of propionic acid and with some of the fraction mentioned above, boiling between 125°–150°, give precisely similar results when submitted to this test, a white powdery salt being precipitated from each.

The three fractions obtained from the lower boiling portions of the acids obtained from the fermenting *ulva* were examined as follows.

A roughly graduated pipette was made, and with it the same volume—about 0·2 c.c.—of each of the fractions was removed, weighed, diluted with water, titrated with $\frac{N}{10}$ baryta, the titrated fluid evaporated to dryness, and heated at 100° C. until of constant weight, then ignited with sulphuric acid, and the resulting barium sulphate weighed.

The following table contains the results, calculated in such manner as to be comparable both with each other and with the theoretical quantities required for acetic, propionic, and butyric acid respectively.

Fraction	Acid taken	c.c. $\frac{N}{10}$ Baryta required for 1 part of acid	Weight of dry Barium salt from 100 parts of acid	Percentage of Barium
110°–125° C.	0·2430 grm.	126·1	177·3	50·68
125°–150° ,,	0·2332 ,,	129·1	189·7	48·54
150°–165° ,,	0·2192 ,,	113·7	182·2	44·70
$C_2H_4O_2$	requires	167·0	212	53·72
$C_3H_6O_2$	,,	135·0	191	48·41
$C_4H_8O_2$	,,	113·6	181	44·05

Although these results are not entirely satisfactory or concordant, they certainly point to the occurrence of acetic as well as propionic and butyric acids among the products of the fermentation of *ulva* in sea water, but the identification of the former with certainty was not possible, owing to the smallness of the low boiling fraction and the difficulties attending the separation of a given acid of the fatty series from a mixture with its homologues. But the boiling point and higher specific gravity of the low boiling fraction, as well as the percentage of barium found in the barium salt obtained from it, can scarcely be accounted for except on the assumption that it contained acetic acid.

Our experiments on the fermentation of *Ulva latissima* in sea water thus afford evidence that at least three members of the fatty series of acids are produced: of these, however, propionic acidity formed in by far the largest quantity.

The Composition of the Tissues of Ulva latissima.

The occurrence of the *ulva* in two localities (Dublin harbour and Belfast Lough) in considerable quantities where crude sewage makes its way into the sea, and the experiments just recorded on the products of its fermentation, raised several questions which rendered it advisable to make both an ultimate and proximate analysis of its tissues. Thus, if the growth of the weed in quantity is induced by pollution of the sea water by sewage, the weed itself might possibly be found to contain a higher proportion of nitrogen than is present in other seaweeds which luxuriate only in pure sea water.

Again, as regards the products of its fermentation. What substance present in its tissues gives origin to the propionic and other acids? Is it a carbo-hydrate; and if so, what carbo-hydrate, and whence come the ferrous sulphide and sulphuretted hydrogen which are produced abundantly in the later stages of the fermentation?

Ultimate analysis.—For the ultimate analysis a considerable quantity of the seaweed was collected, and each frond separately washed in tap water, and finally with distilled water. The sea-

weed was then drained, pressed between filter-paper, dried in the air, and then in a desiccator until it was quite brittle, when it was reduced to a fine powder in a mortar, and the powder then dried in a weighing bottle. All the different determinations were made on portions of the same stock of seaweed thus prepared.

Ash.—To obtain the ash, weighed quantities of the weed were ignited in a platinum crucible until the residue was of constant weight. We give below all the results obtained, but may remark that (2) and (3) are probably too low, from loss of sodium or potassium chloride.

(1) 0·6502	grm.	yielded	0·1001	grm.	ash	= 15·39%
(2) 0·4582	,,	,,	0·0667	,,	,,	= 14·56 ,,
(3) 0·4753	,,	,,	0·0698	,,	,,	= 14·68 ,,
(4) 0·4958	,,	,,	0·0762	,,	,,	= 15·37 ,,
(5) 0·3248	,,	,,	0·0499	,,	,,	= 15·36 ,,
					Mean	15·07%

Total Nitrogen.—by Dumas' method.

(1) 0·5280 grm. gave 27·6 c.c. nitrogen at 16° C. and 764 mm. = 26·21 c.c. at N.T.P. = 0·032836 grm. = 6·22%.

(2) 0·1986 grm. gave 10·5 c.c. nitrogen at 19° C. and 764 mm. = 9·87 c.c. at N.T.P. = 0·01237 grm. = 6·23%.

(3) 0·7122 grm. gave 36·6 c.c. nitrogen at 17° C. and 764 mm. = 34·63 c.c. at N.T.P. = 0·043382 grm. = 6·09%.

Mean = 6·18 ,,

Carbon and Hydrogen.—The powdered weed was ignited in a closed combustion tube with chromate of lead.

(1) 0·6127 grm. gave	0·2937 grm.	H_2O = 0·0326 grm.	H =	5·33%.
	0·7936 ,,	CO_2 = 0·2164 ,,	C =	35·32 ,,
(2) 0·6560 ,, ,,	0·3075 ,,	H_2O = 0·0342 ,,	H =	5·21
	0·8413 ,,	CO_2 = 0·2294 ,,	C =	34·98 ,,

Mean % of hydrogen = 5·27

,, carbon = 35·15

The composition of the tissues of *Ulva latissima* deduced from the preceding analysis is—

Carbon,	35·15		
Hydrogen,	5·27		
Nitrogen,	6·18		
Oxygen (by difference),	38·33		
Ash,	15·07	containing	Sulphur, 3·21; Iron, 2·20
	100·00		

Proximate analysis.—The attempts which we have made to isolate any definite compounds as proximate constituents of the *ulva* have not been very successful, but it is only fair to ourselves to say that we have not had time to study the matter exhaustively.

Various experiments were tried with different solvents.

When boiled with water the seaweed does not soften nor suffer apparently any considerable change, and no blue colour is produced when the infusion is treated with iodine.

A special experiment was made to ascertain whether any carbohydrate was present capable of ready hydrolysis into a sugar, and for this purpose a quantity of the washed *ulva* was treated for a week in the cold with water containing 5 per cent. of sulphuric acid. The extract was then drained away from the seaweed, excess of barium carbonate added, the solution filtered and evaporated.

During the evaporation, white amorphous matter separated and oily globules also. The dried residue was treated with a little water, the solution filtered and heated on a water-bath with 2 grms. of crystallised phenyl-hydrazine hydrochlorate, and 3 grms. of sodium acetate, but no trace of a crystallised osazone could be obtained. For the sake of comparison, a mixture of 1 grm. of ordinary dextrose with the same quantities of phenyl-hydrazine hydrochlorate and sodium acetate and water was heated and gave abundance of the yellow osazone.

The amorphous matter turned out to be magnesium carbonate, with practically no organic matter.

The remainder of the *ulva*, after treatment with acid, was well washed with distilled water, and then digested in the cold for a week with 5 per cent. caustic soda solution. The resulting brown

liquid was coloured green by a slight excess of hydrochloric acid, and a brownish flocculent precipitate was produced.

By treating the *ulva* with alcohol or ether, the chlorophyll, etc. are only very slowly dissolved. In a preliminary experiment some of the dry seaweed was submitted to the boiling reagent in an extraction apparatus for a week—ether first, and alcohol later—but at the end of that time it was still green in parts.

In a later experiment, 14 grms. of the *ulva*—washed, dried, and roughly powdered—were boiled in a flask with inverted condenser for a week with alcohol. Each day the alcoholic extract was filtered off and distilled from the same (tared) flask, the distillate being again employed for the extraction.

The dried alcoholic extract weighed 2·35 grms., or about 17 per cent. of the weight of the original dried seaweed.

What remained of the latter was then dried and digested in the cold for eight days with a 5 per cent. solution of caustic potash. The liquid was then filtered off through a weighed filter, and the residue of seaweed collected on the latter, well washed and weighed. It amounted to about 7 grms.

On the supposition that alcohol removed all the chlorophyll, fat, etc., and the caustic potash the albuminoids, the composition of the dried seaweed may be represented thus:—

Chlorophyll, fats, etc., . . .	17%
Albuminoids or 'proteids,' . .	33 ,,
Cellulose, ...	50 ,,
	100%

If the percentage amount of nitrogen found in the *ulva* be multiplied by the factor 6·25 (often employed for calculating in such cases the 'Proteids'), the result is 38·6, which is not very different from 33, and it must be remembered that the experiment was only roughly quantitative.

Bacteriological Examination.

From the chemical examination of the products of the fermenting *ulva*, it seemed probable that it was attacked by at least two

different species of micro-organisms,—the first producing fatty acids together with hydrogen and carbonic anhydride; the second causing the formation of sulphides.

The evidence on this point was tolerably clear, for on several occasions no sulphides were produced at all, and, as a consequence, no blackening of the weed occurred, and no evolution of sulphuretted hydrogen, although fermentation had been active, and fatty acids had been plentifully produced, together with hydrogen and carbonic anhydride. And in all our experiments in which the weed blackened, the acid-producing phase of the fermentation preceded that of the sulphide formation by a considerable interval.

Also, when the *ulva* was allowed to ferment in tap water and not in sea water, the production of sulphides was always delayed, and very often did not occur at all.

Numerous attempts have been made to isolate the organisms causing the two changes, but not with absolute success; and we may take this opportunity to express our thanks to Dr Lorrain Smith and Dr Tennant for the assistance they have given us in this branch of the investigation.

Stained preparations of the fermenting *ulva* showed that spore-forming bacilli similar in appearance to *B. tetani* were abundant, but all attempts to isolate them by Koch's plate method or Esmarck's roll tube (anaerobic) cultures, either with ordinary gelatine or agar media, failed, practically no colonies appearing.

A special culture fluid was then made with sea water containing 1 per cent. peptone and 1 per cent. glucose, and (after sterilisation) flasks of this were inoculated (A) with a droplet of the liquid from a tube containing fermenting *ulva* and sea water, and (B) with a minute fragment of the *ulva* itself from the same tube after its contents had been heated for twenty minutes to 80° C., to destroy all but spores.

These cultures when incubated grew, and showed, it was thought, some signs of gas evolution.

After five days agar plate cultivations were made from both, but no colonies appeared. Similar cultivations were made with a medium containing 1 per cent. peptone and 1 per cent. glucose with sea water and agar, both under aerobic and anaerobic conditions, but again without obtaining any definite growth of colonies.

On the other hand, the glucose peptone sea water medium which had been inoculated with a fragment of the fermenting weed, heated to 80° C., developed acid, the amount of which was determined by $\frac{N}{10}$ baryta solution.

10 c.c. after 3 days' incubation at 37° C. required 2·2 c.c. $\frac{N}{10}$ baryta = 0·01628 grms. propionic acid.

10 c.c. after 7 days' incubation at 37° C. required 2·68 c.c. of $\frac{N}{10}$ baryta = 0·0198 grms. propionic acid.

An experiment was then made on a larger scale with this culture fluid, which was sterilised and inoculated with some drops of the liquid from a test-tube containing sea water, glucose, and peptone, and a fragment of the fermenting *ulva*.

The flask containing the inoculated fluid was provided with a cork and an arrangement for collecting any gases which might be evolved, and was placed in an incubator, where it remained for two or three weeks, but no appreciable quantity of gas came off. The contents of the flask were then distilled with sulphuric acid, the distillate boiled with excess of barium carbonate, and the filtered solution evaporated to dryness. A small quantity of a gummy barium salt remained, which qualitatively resembled the crude barium salt obtained from the fermentation of the weed, but its amount was not sufficient for any quantitative experiments.

Attempts were next made to obtain colonies of the micro-organism by employing a substratum of the weed itself. Some fronds of the *ulva* were pressed and dried, and then attached to glass plates by weak gelatine solution. The plates so prepared were next sterilised by heat, and three of them treated as follows:—

On No. 1, some sterilised gelatine solution was poured, previously inoculated with a droplet taken from a tube containing sea water and a fragment of *ulva* which had fermented but had not blackened. On No. 2 some sterilised agar was poured which had been similarly inoculated; and on No. 3 the same medium, inoculated from the same source, which had previously been heated to 80° C. for twenty minutes.

Of these three plate cultivations, well-marked colonies appeared on Nos. 1 and 2. No. 3, was doubtful and too much dried up.

Two colonies from No. 1 and four from No. 2 were planted out in tubes containing fragments of *ulva* and sea water previously sterilised. In three days the tubes inoculated with No. 1 had given off a good deal of gas, and one of those inoculated with No. 2 had also given off gas and its contents were turbid. It seems probable, therefore, that by this method the organism causing the acid fermentation was isolated.

Regarding the second or sulphide-forming phase of the fermentation, as we have already said, it always occurred much later than the first or acid phase, and frequently did not take place at all; and although the presence of sea water does not appear to be absolutely essential to its occurrence, yet undoubtedly it materially assists it, and for that reason we are inclined to believe that the sulphides owe their origin chiefly to sulphates in the water, and possibly in the *ulva* itself, and not to the albuminoids present in such abundance in the weed.

There is also some evidence to show that the organisms concerned in the process occur in the *mud* of the foreshore where the *ulva* is found, and not in the sea water.

The following experiment brings out these facts.

A number of test-tubes were partly filled with sea water, and others with tap water, and in each a piece of *ulva* was placed previously well washed in tap water, and all air bubbles adhering to the weed were got rid of by pressure with a glass rod.

A cotton-wool plug was then inserted in the mouth of each tube, which was made to support a strip of paper moistened with lead acetate, which hung about an inch above the surface of the liquid.

In addition to five such tubes containing the washed weed and sea water, and five containing the washed weed and tap water, two similar tubes were prepared containing *unwashed* weed, one with sea water and the other with tap water. All the tubes were then placed in the incubator. In 24 hours the acid phase of the fermentation had commenced in all the tubes, indicated by the inflation of the weed by the evolved gases.

In 48 hours the lead paper in the tube containing the *unwashed* weed and sea water had begun to blacken distinctly, and that containing the *unwashed* weed and tap water was also tinged, though faintly. In 72 hours the lead papers in all the sea water

tubes were strongly blackened, but those in the fresh water tubes remained unaffected, except the one in the tube containing the unwashed weed. In 168 hours the unwashed weed in sea water was itself beginning to blacken, but the contents of the tap water tubes had still only faintly blackened their lead papers. Even after a month, the difference in the appearance of the contents of the two set of tubes was very noticeable.

Zelinsky * has described an organism which he named *Bacterium hydrosulfureum ponticum*, and obtained from the ooze of the Black Sea, which reduces sulphates to sulphides, and evolves sulphuretted hydrogen. He employed a special culture fluid for its growth, containing 1 per cent. solution of ammonium tartrate, 1 to 2 per cent. solution of grape sugar, $\frac{1}{2}$ to $\frac{1}{3}$ per cent. of sodium hyposulphite, 0·1 per cent. of potassium phosphate, and traces of calcium chloride.

We prepared some of this fluid and inoculated (sterilised) tubes of it with minute fragments of the following :—

1. *Ulva* which had fermented with sea water for 12 months in a stoppered bottle.

2. *Ulva* which had been fermented with tap water for the same time and under similar conditions.

Lead papers were suspended in the upper part of each tube by cotton-wool plugs, and the tubes then placed in the incubator. We also prepared a similar set of tubes containing $\frac{1}{3}$ per cent. of ferrous sulphate instead of the hyposulphite, and inoculated them in the same way. The first series we shall call A and the second B.

In 99 hours the lead papers in all the tubes were blackened except B 2, and a filmy growth was beginning to form on the surface of the liquid in two of the tubes. After a further interval of 24 hours, A 1 was covered with a pink growth, and A 2 with a white growth. A 1 was plated out in ordinary agar medium, but it gave no colonies, but A 2, similarly treated, gave plenty of well-defined colonies. Three of the latter were again plated out and inoculation from the resulting colonies made in tubes containing sterilised *ulva* and sea water, when after 5 days a whitish growth began to appear in the tubes, and 2 days later their contents were giving off sulphuretted hydrogen.

* Zelinsky, *Proceedings of the Russian Physical and Chemical Society*, 25, fasc. 5 [1893].

We have not had time to pursue the bacteriological investigation further, which very possibly in more experienced hands might have given more definite results, but we believe that the following conclusions are warranted from our experiments:—

(1) When the *Ulva latissima* ferments in water, it is attacked by a species of micro-organism, which is a spore-forming bacillus, and which probably infests the weed itself. The products of this fermentation consist mainly of propionic acid, but other fatty acids are formed in smaller quantities, together with carbonic anhydride and hydrogen. This micro-organism *probably* attacks the albuminoids of the seaweed.

(2) The fermenting *ulva* is attacked later by a second species of micro-organism, with the production eventually of ferrous sulphide and sulphuretted hydrogen. It seems probable that these sulphur compounds are produced from the sulphates of the sea water (or those contained in the tissues of the *ulva*), and not from the albuminoids of the seaweed, and that the micro-organisms are derived from the mud of the foreshore where the *ulva* grows.

Our experiments so far do not enable us to decide definitely whether the sulphuretted hydrogen is produced *directly* from the sulphates (or possibly the albuminoids), or *indirectly* from the ferrous sulphide, by the action of the organic acids. We are, however, of the opinion that some of the gas at least owes its origin to the second of these two causes.

Ulva latissima *in relation to Sewage Pollution.*

The evidence which we have collected tending to prove that the occurrence of *Ulva latissima* in quantity in any locality is associated with the pollution of the sea water by sewage is of three kinds.

First, that afforded by the composition of the weed itself, or rather by the proportion of nitrogen it contains. Second, from experiments made on the assimilation of nitrogenous compounds by the growing *ulva* from sea water purposely polluted; and third, from an examination of the localities in which the weed occurs in abundance, and of those from which it is virtually absent.

We shall discuss each of these separately.

1. *The proportion of nitrogen in the* Ulva.—The most interesting and important result of our analyses of the tissues of the weed—and certainly the most surprising one to us—is the extraordinary proportion of nitrogen which they contain. In the following table the percentage of nitrogen in some other (dried) seaweeds is compared with that of the *ulva*, as well as the 'proteine,' obtained by multiplying the percentage of nitrogen by the factor 6·25.

	Percentage of Nitrogen.	Percentage of Proteine.
Ulva latissima,	6·18	38·625
* *Chondrus crispus* (Carragheen Moss),		
Bleached, from Bewlay Evans,	1·534	9·587
,, second experiment,	1·485	9·281
Unbleached, from Ballycastle,	2·142	13·387
,, second experiment,	2·510	15·687
* *Gigartina mamillosa*, from Ballycastle,	2·198	13·737
* *Laminaria digitata*,	1·588	9·925
* *Rhodymenia palmata* (Dulse),	3·465	21·656
* *Porphyra laciniata*,	4·650	29·062
* *Sarcophyllis edulis*,	3·088	19·300
* *Alaria esculenta* (Murlins),	2·424	15·150
† *Fucus saccharinus*,	2·29	...
† ,, *digitatus*,	1·46	...
† ,, *vesiculosus*,	1·57	...
† ,, *ceramium rubrus*,	2·03	...

Not only is the proportion of nitrogen in *ulva* extraordinarily high compared with that present in other seaweeds, but also with vegetables generally. Indeed, in nitrogen content it resembles an animal rather than a vegetable product, as will be seen from the subjoined list of a few typical substances:—

	Percentage of Nitrogen about:—
Animal.	
Meat (dry),	$10\frac{1}{2}$
Cheese (dry),	7
Milk (dry residue),	5
Vegetable.	
Peas,	4–$4\frac{1}{2}$
Clover Hay,	3
Wheat,	$2\frac{1}{2}$
Meadow Hay,	2

* Thorpe's *Dict. Appl. Chem.*
† Würtz, *Dict. d. Chim.*

The farmers on the shores of Belfast Lough have discovered the value of the *ulva* as a manure, and large quantities are carted away by them and used on their land. It no doubt fails in phosphates, but contains the necessary potassium salts. It is probable that it would be greatly improved for most crops by the addition of calcium phosphate or basic slag.

2. *Assimilation experiments.*—Our first idea was to contrast the extent of growth of the *ulva* in pure sea water and in polluted sea water respectively; and accordingly, as far as possible, similar tufts of the growing weed, adhering to stones, etc., and freshly removed from the sea-shore, were placed in two glass aquaria, one of which was filled with the pure sea water of the Irish Channel and the other with the same water to which $2\frac{1}{2}$ per cent. of Belfast sewage had been added. Photographs were then taken of the two tanks, with the object of contrasting them with photographs on the same scale to be taken later.

The result of this experiment was, however, curious, as the weed in both tanks soon became unhealthy and died in a month or two. We believe that the explanation was that in both cases the *ulva* was killed by the strong sunlight to which at times it was exposed, as the two aquaria containing it were placed in a window facing south-west, and the experiment was made in the spring-time.

Since then we have had a specimen of the seaweed growing in a glass dish placed near a window with a northern aspect; for months, and as we write, it is still in a perfectly healthy condition. It is a mere frond of the *ulva*, and was picked up on the shore, unattached to any support, and indeed, when we commenced our experiments with it, we were afraid that it would be of no use to us. But this frond has remained in perfect health for seven months, during which time it has been treated with several different specimens of foul sea water, and in the intervals has not been supplied with any pure sea water; for after the failure of our first experiment it occurred to us that a far better method of investigation would be to examine the water in which the weed was growing, and not the weed itself.

The methods of water analysis are delicate, and by contrasting the composition of samples of sea water both before and after the

ulva had been allowed to grow in them, it seemed to us that the information we desired ought to be readily obtained.

The following experiments were therefore made:—

The frond of *ulva* employed was well washed for about an hour in running tap water to free it from debris. It was of large size, its area being 147 square inches, or about 1 square foot, and its active surface therefore twice that amount.

The dish in which it was placed was a circular glass vessel, with flat bottom and vertical sides, 8 inches in diameter and 3 inches high. It was provided with a cover similar to itself, and it contained in our experiments 1600 c.c. of water. In order to get the frond of seaweed into the dish, it was folded across the middle.

Experiment 1.—*Assimilation of Ammonia.*—A sample of sea water was employed which was collected from a locality in Belfast Lough where several small sewers discharge directly into the sea, and was therefore presumably polluted. The frond of *ulva* was rinsed in the dish with some of this water, which was thrown away, and the dish then filled with more of the same water, the air-bubbles entangled in the folds of the seaweed being got rid of by gentle pressure with a glass rod. Some of the water was analysed before this was done, while after a week had elapsed a quantity of the water was removed from the dish and also analysed.

The results of the two analyses were as follows:—

	Parts per 100,000.	
	Free Ammonia.	Albuminoid Ammonia.
Original sea water, ...	0·046	0·020
After contact with the *ulva* for 7 days, ...	0·000	0·020

The seaweed had therefore absorbed *every trace of free ammonia* from the water, a result which was quite unexpected and highly interesting. On the other hand, none of the albuminoid matter had been absorbed, which, however, is quite in accordance with the known facts regarding plant nutrition. In order to verify this result, the *ulva* was allowed to remain in the same water for another week, when a second analysis was made with precisely the same results as before.

Experiment 2.—*Assimilation of Ammonia.*—This experiment was made with the view of getting some idea of the rapidity with which the *ulva* can absorb free ammonia from sea water, and also to ascertain whether it can thrive in a very highly polluted water. The frond of *ulva* had remained in the sea water of the last experiment an additional four days, making eighteen days in all, and appeared to be quite healthy.

A sample of sea water was collected from the same locality as before, and to it 1 per cent. of sewage was added (obtained from the pumping station at the Belfast Main Drainage Outfall). This mixture gave, on analysis, the following results:—

	Parts per 100,000.	
	Free Ammonia.	Albuminoid Ammonia.
Sea water *plus* 1 per cent. sewage,	0·030	0·024

But as it contained less free ammonia than was expected, a standard solution of ammonium chloride was added, sufficient to bring up the free ammonia to 0·050 parts per 100,000.

The frond of *ulva* was drained from the first sample of sea water and rinsed with this mixture, again drained, and the dish then filled with the same mixture.

It was intended to make a series of analyses of the contents of the dish at intervals of about twenty-four hours, but to our astonishment we found that practically the whole of the free ammonia had been absorbed after a period of only seventeen hours, as the following determination shows:—

	In 100,000. Free Ammonia.
Sea water, *plus* sewage and ammonium chloride, after contact with the *ulva* for 17 hours, .	0·001

With the object of tracing the fate of the albuminoid matters, the weed was allowed to remain for about five weeks in contact with the mixture, while a flask containing the same mixture was also kept during the same interval. Both samples were then analysed, with the following results:—

	Parts per 100,000. Free Ammonia.	Parts per 100,000. Albuminoid Ammonia.
Sea-water, *plus* sewage and ammonium chloride, kept for 5 weeks, ...	0·050	0·016
The same mixture after contact with the *ulva* for 5 weeks, .	0·004	0·017

These results further prove that the *ulva* cannot absorb albuminoid matters.

Experiment 3.—*Assimilation of Nitrates.*—The result of these experiments, as well as other considerations to be mentioned presently, induced us to extend our inquiry somewhat further, in order to ascertain whether the *ulva* can absorb nitrogen in the form of nitrates, with the same ease and rapidity as it assimilates that element as ammonia.

The same frond of *ulva* was again used, which had now been under observation in the dish for six months. In the interval from the last experiment, the water in which it was growing had been changed only once. On examination, it was found that, owing to the inflation of some of its under surface by evolved oxygen, a portion of the frond had become quite dry and almost bleached. We thought it highly probable that it was no longer in a sufficiently healthy condition for further experiment; but having no other specimen at hand, we decided to test its vigour by its power of absorbing ammonia.

Some fresh sea water was therefore obtained and examined as follows:—200 c.c. were distilled until 100 c.c. had passed over, and 50 c.c. of this distillate were Nesslerised for the free ammonia. The residue left in the distilling flask was then diluted to the original volume (200 c.c.) with ammonia free water, and treated with zinc-copper couple for twenty-four hours at ordinary temperatures, then poured off, again distilled, and the distillate Nesslerised. The results of the analysis were as follows:—

	Parts per 100,000 Nitrogen, as:— Free Ammonia.	Parts per 100,000 Nitrogen, as:— Nitrates.
The sample of sea water contained,	0·005	0·006

Sufficient ammonium chloride was added to this sea water to bring up the nitrogen as free ammonia to 0·042 parts per 100,000, and this strength was verified by a determination made as before.

The frond of *ulva* was now drained from the sea water in which it had been immersed for some months, rinsed with the new sample prepared as just described, and the dish then filled with the latter.

After twenty-four hours, 200 c.c. of the water were removed from the dish, distilled, and the distillate Nesslerised, when *no* free ammonia was found, proving that the seaweed was still in a perfectly vigorous condition. This was also shown by the copious evolution of oxygen which had occurred from it, the gas remaining entangled in the folds of the frond.

The water in the dish was next poured off, and sufficient of a standard solution of potassium nitrate added to it to bring up the nitric nitrogen to 0·05 parts per 100,000, when it was emptied back again. The frond of *ulva* was now in contact with ammonia-free sea water containing nitrates, and was allowed to remain thus for 70 hours, when a portion of the water was removed from the dish, and the nitrates determined by the same process as before. The water was found to contain 0·005 parts of nitric nitrogen, showing that the *ulva* had absorbed 90 per cent. of the amount originally present.

The results of the preceding experiment leave no doubt as to the energetic power which *Ulva latissima* possesses of absorbing nitrogen from polluted sea water, both in the form of ammonia and of nitrates. They also clearly demonstrate that this seaweed can flourish in highly polluted water; and in addition, they lend a good deal of support to the theory which we had gradually been led to form, that the occurrence of the *ulva* in quantity in a given locality may be regarded as a sign of sewage pollution.

From the results of these experiments it is possible to calculate the rate of growth of the *ulva* under the existing conditions; for, as its tissues contain 6·18 per cent. of nitrogen, it is obvious that the nitrogen lost by the water in which it was placed, multiplied by the factor $\frac{100}{6 \cdot 18}$, gives the weight of the seaweed formed.

Thus, in experiment 2, the water lost 0·049 per 100,000 of free or saline ammonia in 17 hours. This is equivalent to 0·0404 parts of nitrogen per 100,000; and as it was removed from 1600 c.c. of

water, its actual weight was $0{\cdot}0000404 \times 16 = 0{\cdot}0006464$ grm., and this, multiplied by the factor $\frac{100}{6{\cdot}18}$, gives 0·0104 grm., or about 1 centigram, as the actual weight of seaweed formed.

A series of determinations showed that 1 square inch of the dried *ulva* weighs on an average 0·009 grm., so that in this experiment $\frac{0{\cdot}0104}{0{\cdot}009}$, say, 1·1 square inches, of the *ulva* were formed, which is equivalent to nearly 0·8 per cent. of the original frond.

We hope to make further experiments in order to ascertain whether the rate of nitrogen assimilation is constant, or varies with the concentration, and also to what extent the rate is affected by differences in illumination.

3. *The localities in which* Ulva latissima *occurs in quantity contrasted with those from which it is virtually absent.*—We may first of all draw attention to two particular localities which have come more immediately under our observation where this seaweed is abundant, and one from which it is almost entirely absent, because an examination of the conditions obtaining in these, offers some very striking evidence in favour of the view mentioned above, viz., that the occurrence of the *ulva* in quantity is an indication of sewage pollution.

The first two localities we refer to are Belfast Lough and a part of Dublin Bay, and the second is Strangford Lough.

Belfast Lough.—According to the statements of some of the older inhabitants of the neighbourhood, *Ulva latissima* was not present in former times in the very large quantities in which it now occurs in the upper reaches of the Lough, but the *Zostra marina*, or sea grass, now found only in small quantities, was abundant.

Up to the year 1889 the bulk of the sewage of the city of Belfast was allowed to flow directly into the Lagan river. But in that year a new main drainage system was inaugurated by which the greater part of the sewage is collected in two main channels, and from them pumped into a tank, the contents of which are discharged (on the ebb-tide only) through a submarine culvert opening some distance seawards. Belfast, as every one knows, has grown with remarkable rapidity, and there can therefore be no question that for that reason alone very much more sewage makes its way into the Lough now than formerly, and this amount has

undoubtedly been increased since the introduction of the main drainage scheme, the Lagan river no longer acting as a settling-tank in which the bulk of the sewage solids were deposited.

The tides in the upper reaches of the Lough are sluggish, and from float experiments made by the engineer to the Harbour Board, it would seem that the greater part of the sewage does not make its way out of the Lough on the ebb-tide, but having drifted a certain distance seawards, is washed backwards by the flood-tide in a bifurcating stream, which distributes it over a wide area.

In Dublin Bay the conditions under which *Ulva latissima* occurs in quantity are both interesting and significant.

Broadly speaking, the upper reaches of the Bay are divided artificially into two portions by the so-called Pigeon House wall, which extends for more than a mile and a half in an easterly direction, and terminates in Poolbeg lighthouse. A second wall, called the North Bull wall, juts out from the northern shore of the Bay at Dollymount, and extends in a S.E. direction to within about 1000 feet of Pool Beg lighthouse, terminating in a second lighthouse called the Bull. The northern part of the Bay thus almost enclosed by the two walls forms the harbour. On the other hand, the southern portion of the Bay is quite open.

The harbour receives not only the waters of the Liffey river into which the major portion of the city sewage at present flows, but also those of the Tolka river, which is polluted by a large sewer running into it close to its mouth, while another large sewer discharges directly on to the northern shore close to the city, as well as a considerable number of smaller sewers the whole way thence to Dollymount.

On the other hand, no sewers of any magnitude (if indeed any at all?) discharge their contents into the southern portion of the Bay until Blackrock and Kingstown are reached, which are quite at its mouth. Thus, broadly speaking, the northern portion of Dublin Bay consists of a polluted area, while the southern portion is unpolluted. Now, plenty of the *ulva* is found on the northern shores of the harbour, and is washed up along the Clontarf foreshore, where, as in Belfast Lough, it rapidly putrefies in warm weather, and gives rise to a nuisance. On the other hand, the southern portions of the Bay seem to be quite clear of the sea-

weed until Blackrock and Kingstown are reached, where it is found in fair quantity.

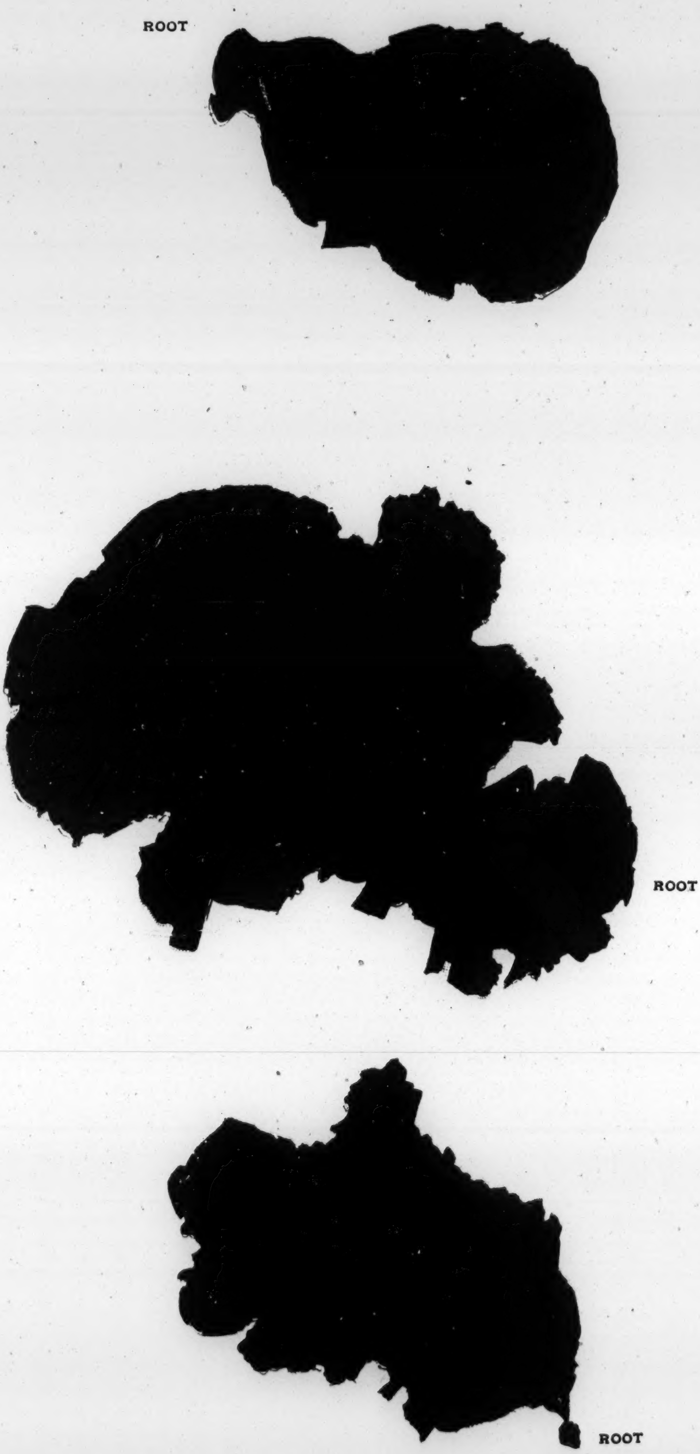
On Plates 2 and 3 we give charts of these two localities (Belfast Lough and Dublin Bay), on which we have marked in black those areas over which the *ulva* is chiefly distributed. It must be recollected that much of this seaweed is unattached by any stalk, and drifts about from place to place. Hence no chart can be drawn to represent where it will be found on all occasions, and the Plates must therefore, in respect of the occurrence of the weed, be regarded merely as diagrams.

Strangford Lough, which is quite close to Belfast Lough, resembles the latter in extent of area, and also in the large surfaces uncovered in its upper reaches at low water. It differs from it, however, in that no large town is situated on its banks. In this Lough *Ulva latissima* is practically absent.

The above-mentioned facts seem to offer strong *prima facie* evidence that the growth of *Ulva latissima* is associated with sewage pollution of sea water, and as a consequence that its occurrence in quantity in a particular locality may be regarded as an indication of sewage pollution. There can, at all events, be no doubt as to the nuisance which this seaweed can at times give rise to, which closely resembles that proceeding from very foul sewage. And there can also be no doubt as to the extraordinary powers of nitrogen assimilation which it possesses.

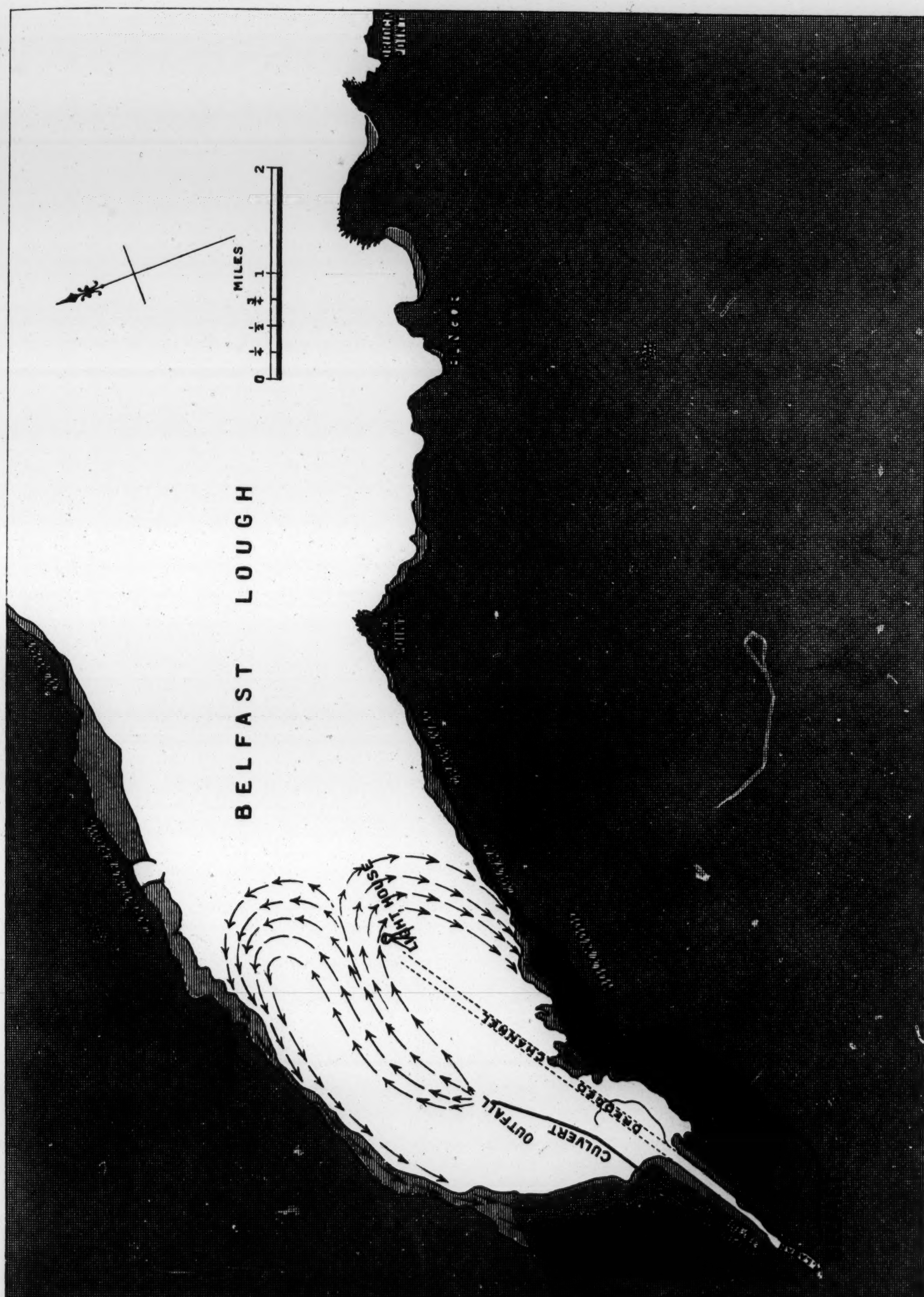
ULVA LATISSIMA—YOUNG FRONDS.—PLATE I.

(PRESSED SPECIMENS.)



A. RITCHIE & SON EDIN.

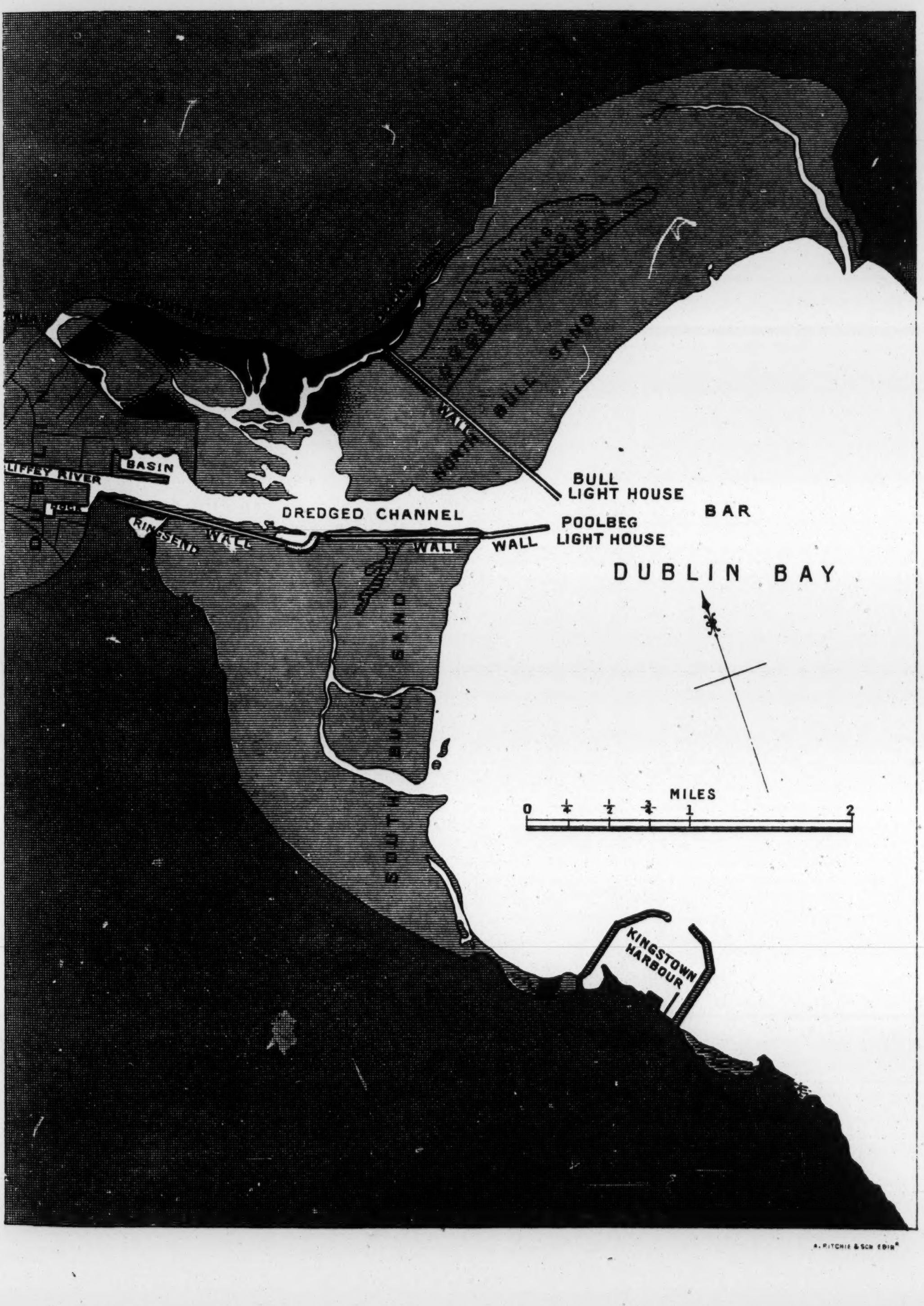
ULVA LATISSIMA IN RELATION TO SEWAGE POLLUTION.



A. Ritchie & Son, Edin.r

Plate 3.

ULVA LATISSIMA IN RELATION TO SEWAGE POLLUTION.



A. Ritchie & Son Edin.

EXPLANATION OF PLATES.

Plate 1.

Young plants of *Ulva latissima* with root attachment as they appear when pressed.

Plate 2.

Diagram to illustrate the occurrence of *Ulva latissima* in Belfast Lough.

The *light* shading indicates the shore or banks uncovered at low water.

The *dark* shading indicates those parts of the shore or banks uncovered at low water where the *ulva* abounds. The arrows indicate the distribution of sewage on the ebb and flood tides.

Plate 3.

Diagram to illustrate the occurrence of *Ulva latissima* in Dublin Bay.

The *light* shading indicates the shore or banks uncovered at low water.

The *dark* shading indicates those parts of the shore or banks uncovered at low water where the *ulva* abounds.

Solar Radiation and Earth Temperatures. By Professor C. G. Knott. (With a Plate.)

(Read January 21 and February 4, 1901.)

At a recent meeting of the Society, Dr Buchan read a paper based on certain observations of the temperature of the waters of the Mediterranean, which had been made by the staff of the Austrian ship *Pola*. These indicated that the direct effect of solar radiation was felt to a depth of over 150 feet. At any rate, the facts were that the temperature of the upper stratum of water of this thickness was perceptibly higher at about 4 p.m. than at 8 a.m., and that the difference was about 1°·5 Fahr. or 0°·8 Cent. at the surface, diminishing fairly steadily to value zero at a depth of fully 150 feet or 50 metres. It may easily be calculated that this excess of temperature at the afternoon hour means the accumulation of an amount of heat equal to 1460 units in every column of water 1 square centimetre in section; and this is accomplished within the eight hours from 8 a.m. to 4 p.m. It must be noted that this accumulation of heat is a daily occurrence.

The whole process of the heating and cooling of any portion of the earth's surface is a very complicated one. Doubtless there is constant radiation into space going on steadily day and night. During the day the solar energy enters the atmosphere and part of it reaches the earth's surface, heating the matter there. At night this direct heating effect is absent. There must, therefore, result a steady periodic state of temperature change, a daily see-saw, as much on the average being lost every night as is gained every day. This daily fluctuation is of course subject to a seasonal variation, depending primarily on the declination of the sun, but also, as Langley has shown, on atmospheric conditions, the true nature of which is at present a matter of speculation. But whatever these conditions may be, and whatever may be the real physical process by which the see-saw of temperature is produced in the Mediterranean waters, we must regard this resultant accumulation of heat during the day as due to solar radiation, direct and indirect. And the first question which demands an answer is, what fraction of the whole heat supplied by the sun is repre-

sented by this quantity which gets stored up in the surface waters of the Mediterranean? Making a rough calculation, I found that this stored-up heat was more than could be reasonably accounted for if we accept Langley's estimate of the solar constant. According to Langley's measurements, the solar energy which flows every minute normally across a square centimetre of the earth's surface, after a portion has been absorbed by a clear atmosphere, is about 2 calories. In other words, if a cubic centimetre of water were set with one face pointing to the sun, and if the solar energy crossing that face were all transformed into heat within the cubic centimetre of water, the temperature of the water would be raised 1° Cent. in one minute. Hence an accumulation of 1460 calories under each square centimetre of the surface means that with a steadily vertical sun, and with no loss in other directions, the sun would require to shine for 590 minutes, or nearly six hours. But six hours of a vertical sun is an impossibility, and it is certain that the solar radiation incident upon the face of the waters is not wholly transformed into heat within the water. A definite fraction is reflected, and a definite amount must always be passing out by convection, radiation, emission, and other processes. Taking all these conditions into account, we have great difficulty in believing that, between the morning and afternoon of each day, heat to the amount of 1460 units can be accumulated in the surface waters of the sea, unless we can discover some other source of heat than the direct radiation of the sun.

To make the comparison more complete, I have made a detailed calculation of the amount of solar heat supplied to each square centimetre of the earth's surface in the latitude of the Mediterranean, the calculation being based on Langley's broad results. To make an accurate calculation is at present an impossibility; for the necessary data are not yet to hand. Langley has shown indisputably that selective absorption in the atmosphere makes it impossible to treat the absorptive action of the air as a whole. That is to say, if the radiant energy of the sun is reduced from E to aE after transmission through a given mass of air, we cannot assume that it will be reduced to a^nE after transmission through n times the given mass of air. The assumption may reasonably enough be made for each individual ray; but, since the coefficient of trans-

mission varies greatly with the wave-length and according to a law which experiment alone can discover, the use of a mean value of a for the whole radiation will necessarily give too great a value for the transmissibility through increasing masses of air. Bearing this in mind, we may for the present purpose assume the law mentioned, although we know that it is only a first rough approximation and will give too high a value for the transmissibility when the altitude of the sun is small.

Langley's broad result is that the energy of the solar radiation, which reaches the earth's surface after transmission through the vertical depth of atmosphere, is about two-thirds of the energy which would reach the surface if the air were absent. Calling this coefficient of transmission a, we see that if ζ represents the zenith distance of the sun the mass of air traversed is roughly proportional to $\sec\zeta$. The radiation falling normally on unit surface is therefore proportional to $a^{\sec\zeta}$. Hence the radiation falling on each square centimetre of the earth's horizontal surface is proportional to $\cos\zeta . a^{\sec\zeta}$. If we multiply this by the element of time and integrate from sunrise to culmination, we shall get half the quantity of solar energy which falls on each square centimetre of the earth's surface during one day. Let λ be the latitude of the place and δ the sun's declination at the time considered, then the zenith distance ζ is connected with the time by means of the formula

$$\cos\zeta = \sin\lambda\sin\delta + \cos\lambda\cos\delta\cos\omega t$$

where ω is the angular velocity of the earth about its axis.

The evaluation of the integral

$$\int \cos\zeta . a^{\sec\zeta}\, dt$$

can be effected with sufficient accuracy by graphical methods. To this end the quantity $\cos\zeta . a^{\sec\zeta}$ was calculated for a series of convenient values of ζ, and then, by means of the formula given above, the corresponding values of t were calculated for the positions of the sun at intervals of a month, ranging from summer to winter solstice. For each value of the sun's declination a curve was then drawn, the abscissæ of which were the times reckoned from culmination, and the ordinates the corresponding values of

the relative solar radiation falling on unit horizontal surface, the unit radiation being the quantity that would have fallen normally on a square centimetre had there been no atmospheric absorption. The data from which these curves were constructed are given in the following table.

Table showing the time in hours reckoned from culmination at which for given values of the sun's declination, as shown in the top row, the radiation crossing unit horizontal surface in lat. 33° N. *has value as shown in the first column.*

R.	+23° 72′	+20	+12	0	−12	−20	−23° 27′	Sun's decl.
·703	0							Time in hours.
·675		0						
·638			0					
·606	1·83	1·67	1·11					
·549				0				
·512	2·71			1·17				
·427					0			
·333						0		
·331	4	3·88	3·57	2·82	1·96			
·302							0	
·245	4·53						1·46	
·0914	5·51	5·44	5·11	4·6	3·98	3·49	3·21	
·06	5·82							
·0073	6·44	6·28	5·94	5·43	4·86	4·44	4·24	
0	7·06	6·89	6·53	6	5·47	5·08	4·9	

From these seven curves we can estimate the areas, and thus evaluate the integral $\int R dt$ from culmination to sunset or from sunrise to culmination. The results are given in the following small table, in which the first column contains the sun's declination, and the second the relative radiation reaching unit horizontal surface, the unit of time involved being the minute.

Declination.	Half-daily heating (relative).
+23° 27′	158·34
+20	150·57
+12	135·00
0	105·15
−12	73·8
−20	54·0
−23 27	46·8

These numbers are shown graphically in the Plate, fig. 2 (upper curve).

Multiplying the numbers in the second column by twice the value of the solar constant, we get in absolute units the amount of heat supplied daily by the sun to unit area of the earth's horizontal surface. According to Langley's elaborate researches the value of the solar constant may be taken as 3 calories per square centimetre per minute. Hence, multiplying by 6 we find that there fall on each square centimetre of the earth's surface, in the latitude of the Mediterranean, 950 units of heat during the midsummer day.

To compare with the data furnished by the *Pola* observations, which were made during the months of July, August and September, we should however take, not the midsummer value, but the average value during these months. This average is less than 850 units per day. But, further, the temperature observations were made in the morning and afternoon, say, at 8 a.m. and 4 p.m., an interval of only eight hours. Evaluating the areas of the curves through an interval of four hours from culmination instead of through the half day, we get in place of the first four numbers in the small table above the values 136, 131, 120, 97. The mean of these is 121, giving a total supply during the eight hottest hours of the day of only 730 units of heat to each square centimetre of surface.

Let us now consider the data which Dr Buchan has extracted from the *Pola* observations. They are contained in the following table, in which the first row gives the depths in metres, and the second the excess in Fahrenheit degrees of the afternoon temperature over the morning temperature.

Depth, . . .	0	1	2	5	10	20	30	50	75
Temp. Diff. Fahr.,	1°·5	1°·4	1°·3	1°·3	0°·9	0°·5	0°·3	−0°·1	0°

Constructing with these a curve, and estimating the area contained within the curve and the co-ordinate axes, we find, on reducing to Centigrade degrees, that the afternoon excess of temperature means an accumulation during the eight hours of 1460 units of heat under each square centimetre of surface. And yet direct pyrheliometric measurements give us only 730 units of heat

in the same time. We know, moreover, that all the incident solar energy cannot be absorbed by the water, but that a considerable fraction is reflected or escapes in other ways. It therefore seems impossible to explain the afternoon temperature excess down to these depths in the Mediterranean as a result of direct solar radiation. The only other way out of the difficulty is to suppose that there is some considerable error in one or other of the sets of experimentally ascertained facts on which the present discussion is based. To make the facts compatible we should have either to diminish by at least one half the temperature differences observed by the officers and crew of the *Pola*, or greatly to increase the value of the solar constant. I do not think that the broad results obtained by Langley can be seriously called in question, or that there is any ground for believing that the true value of the solar constant can be much greater than the value estimated by him.

A careful study of Langley's measurements and reductions leaves on the mind little doubt as to the main accuracy of his conclusions, which differ from the conclusions of previous investigators by assigning a somewhat greater value to the solar constant. A very careful scrutiny of the conditions under which the *Pola* observations were obtained and the methods employed, supplemented by similar series of observations carried out in wide oceans, might determine how far the results were affected by purely local conditions. At present it seems to be impossible to suggest any satisfactory explanation of the extraordinary magnitude of the depth to which the daily solar radiation apparently penetrates in the Mediterranean Sea.

It has been long known that the solar radiation penetrates to a comparatively small depth in the rocky material of the earth. In 1837 Professor Forbes began a valuable series of observations of temperature at various depths in the rock of the Calton Hill, Edinburgh; and the main conclusions from these may be found in several of our modern text-books (*e.g.* Tait's *Heat*). Thus the conductivity of the rock is easily calculated by methods furnished by Fourier in his classical work *Theorie de la Chaleur* (1822). From this, in combination with the observed rate of increase of temperature with depth, an estimate may be made as to the amount of heat

lost by the earth every year. This is perhaps the most interesting of all results deducible from measurements of earth temperatures.

There is, however, another direction of enquiry suggested by the comparison made in the early part of the present paper, and that is to estimate the accumulation of heat at different times of year throughout the rocky stratum. When this is done a comparison may then be made between the heat so accumulated and the available quantity of energy according to Langley's estimate. Thus we should expect to find that during a particular month of the year there was more heat accumulated in the rocky stratum than during any other month. This will be due to the excess of radiation supplied in the summer months. The relation between these two quantities may possibly lead to an approximate measurement of the emissive power of the earth.

In the calculations which follow I have used as the fundamental data the earth temperatures during the eight years beginning October 1879. These were published by Piazzi Smyth (*Trans. Roy. Soc. Edin.*, vol. xxxv.), and were discussed by him in connection with sun-spot periodicity. There are four thermometers in all, distinguished as Nos. 1, 2, 3 and 4, their depths being respectively 0·8763, 1·4478, 3·238, 6·35 metres. In the following table the mean of the eight monthly means for each thermometer is given for every month throughout the yearly period.

Eight year Means of Earth Temperatures (Fahr.).

	Therm. 1.	Therm. 2.	Therm. 3.	Therm. 4.	Calculated Surface Temp.
October, .	46·445	48·748	48·52	46·863	45·06
November, .	43·785	45·558	47·655	47·136	39·30
December, .	40·284	42·611	46·345	47·146	36·32
January, .	39·859	41·069	44·983	46·908	36·08
February, .	39·28	40·515	43·983	46·521	37·46
March, . .	39·661	40·616	43·414	46·104	39·78
April, . .	41·641	41·628	43·181	45·728	43·30
May, . .	45·108	44·055	43·646	45·450	48·22
June, . .	49·993	47·926	44·863	45·36	53·56
July, . .	52·995	50·78	46·498	45·533	57·00
August, . .	53·12	51·588	47·873	45·896	56·46
September, .	51·48	51·08	48·693	46·443	51·78

The main features embodied in these numbers are indicated in the corresponding curves in the Plate, fig. 1. The well-known manner in which the crest of the temperature wave lags behind as the depth increases is evident at a glance, as also the rapidly-diminishing range of temperature.

Each set of numbers was then treated by harmonic analysis, in accordance with the formula

$$v = A_0 + A_1 \cos\theta + A_2 \cos 2\theta + A_3 \cos 3\theta + A_4 \cos 4\theta + A_5 \cos 5\theta + A_6 \cos 6\theta$$
$$+ B_1 \sin\theta + B_2 \sin 2\theta + B_3 \sin 3\theta + B_4 \sin 4\theta + B_5 \sin 5\theta + B_6 \sin 6\theta$$

where v is the temperature, and the A's and B's constants to be determined by calculation from the twelve linear equations when for each value of the temperature given to v the corresponding value of θ is inserted in the expressions on the right. Beginning with the value of 30° for October, θ increases by 30 in each succeeding month. The constants are tabulated below.

	Therm. 1.	Therm. 2.	Therm. 3.	Therm. 4.
A_0	45·358	45·518	45·8045	46·257
A_1	+5·899	+5·304	+2·672	+0·156
B_1	−4·447	−2·400	+0·728	+0·886
A_2	+0·21	+0·278	+0·2145	+0·0053
B_2	−0·8983	−0·572	−0·048	+0·0462
A_3	−0·1157	−0·125	−0·0408	+0·0047
B_3	+0·3373	+0·227	−0·0055	+0·0107
A_4	−0·0045	+0·0435	+0·0238	+0·0057
B_4	+0·043	+0·0738	+0·0033	+0·0042
A_5	+0·1267	+0·0558	+0·0082	+0·009
B_5	−0·0872	−0·0305	+0·0073	+0·0028
A_6	+0·0123	+0·017	+0·0207	+0·010
B_6	0	0	0	0

Most information is obtained from the first and second harmonic terms in each. According to the recognised theory, it should be possible to combine the first harmonic terms in the formula

$$v = V\epsilon^{-p'x} \cos\left(\frac{2\pi}{T}t - px + q\right)$$

where V is the amplitude at the surface ($x=0$) and p' p q are constants, of which p and p' should have the same value. The constant p' is calculated at once by taking the ratio of any two of the amplitudes, and dividing the Napierian logarithm of this ratio by the difference of depth of the corresponding thermometers. The three values of p' found in this way by combining the 1st and 2nd, the 2nd and 3rd, and the 3rd and 4th, are 0·00436, 0·00386, and 0·00363, giving a mean of 0·00392.

Then p may be calculated from the phases when the expression $A \cos \theta + B \sin \theta$ is thrown into the form $P \cos (\theta + Q)$; for this quantity Q must be equal to $-px+q$. We have four equations to determine two quantities. Working them out by the method of least squares, we find

$$p = 0{\cdot}00371 \qquad q = 0{\cdot}9629.$$

The difference between p and p' is not more than what might reasonably be expected.

Finally, calculating the value of V from each set, we get the four values 10·34, 10·35, 10·03, and 11·2, a very satisfactory result, giving a mean of 10·48.

Hence we may write the most important term representing the annual wave of temperature passing downwards into the rock of the Calton Hill in the form

$$v = 10{\cdot}48\ \epsilon^{-0{\cdot}00392x} \cos\left(\frac{2\pi}{T}t - 0{\cdot}00371x + 0{\cdot}963\right).$$

This gives a wave-length of about 16·93 metres, but before this depth is reached the amplitude of the variation has become too small to be appreciable.

In the expression just given x is measured in centimetres. If, then, we integrate it with regard to dx from x equal to zero to x equal to infinity, and multiply the result by the thermal capacity of unit volume of the rock, we shall obtain an estimate of the quantity of heat which, at a given instant, is contained in the rock per square centimetre of surface. The value is

$$\frac{cV}{p'^2+p^2}\left\{p' \cos\left(\frac{2\pi t}{T}+q\right) + p \sin\left(\frac{2\pi t}{T}+q\right)\right\}$$

where c is the thermal capacity per unit volume.

The greatest positive value of this is when

$$\frac{2\pi t}{\mathrm{T}}+q=\frac{\pi}{4}$$

and the least positive value or greatest negative value is when

$$\frac{2\pi t}{\mathrm{T}}+q=\frac{5\pi}{4}\text{ or }-\frac{3\pi}{4}.$$

The times corresponding to these values are $-0{\cdot}0307$ and $+0{\cdot}4693$ expressed in fractions of a year and reckoning from the middle of September, that is, about the beginning of September and the beginning of March.

Hence there is more heat accumulated within the Calton Hill rock in the month of September than in the month of March by an amount equal to

$$\frac{1}{\sqrt{2}}\cdot\frac{2c\mathrm{V}\ (p'+p)}{p'^2+p^2}=\frac{c\mathrm{V}\sqrt{2}}{p}\text{ approximately}$$

$$=2000\text{ nearly (Fahr. degree).}$$
$$=1111\qquad\text{(Cent. degree).}$$

A better estimate may, however, be made from the temperature observations themselves if we first of all calculate the values at the surface. This requires us to work out the successive harmonics in the same way in which the first has been treated. The results for the second harmonic are as follows. The aim being to express the four harmonic terms in the form

$$\mathrm{V}\epsilon-q'^x\cos\left(\frac{4\pi t}{\mathrm{T}}-qx+e\right)$$

the three values obtained for q' were 0·00659, 0·00592, 0·00497, and the values of q and e worked out from the four-phase relations by the method of least squares were 0·00515 and 1·84. These give 1·656 as the mean value of the amplitude of the temperature variation at the surface.

The comparative smallness of the amplitudes of the third and fourth harmonics, and the shortness of the period of the fifth harmonic, render it quite unnecessary for these to be taken into account. The two harmonic expressions for the surface varia-

tion, obtained from the general expressions by putting x equal to zero, may then be taken as representing fairly well the variation of temperature at the surface. The combined expression is

$$v = 10 \cdot 48\, \epsilon^{-0 \cdot 00397x} \cos\left(\frac{2\pi}{\mathrm{T}} t - 0 \cdot 00371x + 0 \cdot 963\right)$$

$$+ 1 \cdot 656\, \epsilon^{-0 \cdot 00583x} \cos\left(\frac{4\pi}{\mathrm{T}} t - 0 \cdot 00515x + 1 \cdot 925\right).$$

Calculating the numerical values at the surface ($x = 0$) for the successive months, we get a set of temperatures which may conveniently be tabulated along with the means of the observed temperatures at the different depths. We are now furnished with five columns of numbers, each row containing the simultaneous temperatures at the surface and the positions occupied by the thermometers. The calculated values of the surface temperatures are given in the last column of the table on p. 302 above. We may now get fairly accurate determinations of the accumulated heat within the crust at any time by multiplying the mean of the temperatures at each pair of consecutive positions as we descend by the distance between the corresponding positions measured in centimetres. The four quantities so obtained are then added together, and the result multiplied by the thermal capacity per unit volume. Reducing to the Centigrade as unit, and subtracting the smallest of the numbers from all the others, we finally obtain a series of numbers representing the annual gain and loss of heat under each square centimetre of the earth's surface. In this calculation we neglect the heat which penetrates below the deepest thermometer. This, however, is comparatively small, and besides the determination of the surface temperatures will almost certainly involve as large errors. The final results are shown graphically in the Plate, fig. 3, and are given in the following table, which contains, in addition to the monthly values deduced from the temperatures as originally tabulated, intermediate values obtained by calculation from the interpolated values taken from the curves.

Month.	Accumulation of Heat per sq. cm. of Surface.
October,	910 754
November,	604 452
December,	296 183
January,	107 55
February,	18 3
March,	0 27
April,	87 245
May,	330 520
June,	719 909
July,	1041 1128
August,	1189 1212
September,	1161 1045

From these numbers we learn that in the beginning of September there are some 1200 more units of heat under each square centimetre of the Calton Hill than in the beginning of March.

It remains now to compare this accumulation of heat with the amount of energy supplied by solar radiation. To this end we must make for the latitude of Edinburgh the same kind of calculation as was made for the latitude of the Mediterranean in the first part of this paper. The results are given in the following table, drawn up similarly to that on page 299.

Table showing the time in hours reckoned from culmination at which for given values of the sun's declination, as shown in the top row, the radiation crossing unit horizontal surface in lat. 56° N. *has value as shown in the first column.*

R.	+23° 27′	+20°	+12°	0°	−12°	−20°	−23° 27′	Sun's declination.
0·552	0							hours measured from culmination.
·516		0						
·512	1·57	0·81						
·433			0					
·421	2·92	2·43						
·331	3·89	3·48	2·49					
·296				0				
·245	4·7	4·34	3·51	1·77				
·145					0			
·0914					2·03			
·06	6·6	6·2	5·41	4·22	2·63			
·0554						0		
·051							0	
·0073	7·54	7·11	6·27	5·13	3·83	2·61	2·54	
0	8·66	8·11	7·18	6	4·78	3·82	3·79	

From the graphical representations of these seven sets of numbers we can estimate the areas and so evaluate the integral $\int Rdt$ through half a day. With the minute as the unit of time involved, we find the following numbers expressing the relative radiations during half a day for the different declinations of the sun, the unit being the amount that would cross unit area perpendicularly were there no absorption in the atmosphere.

Declination.	Half-daily heating (relative).	Daily heating (absolute).
+23° 27′	141·2	847·2
+20	125·4	752·4
+12	95·5	573
0	51·8	310·8
−12	20·7	124·2
−20	5·48	32·9
−23 27	5·06	30·4

Multiplying the numbers in the second column by twice the

solar constant, namely 6, we get the daily heating expressed in calories. The values are given in the third column.

The particular values of the declination entered in the first column are the values at equal intervals of a month. With these as abscissæ, and with the corresponding values of the energy supplied per day, we may construct a curve showing the manner in which the heating effect varies from day to day throughout the year. The curve is given in the Plate, fig. 3. From this curve by estimation of areas we can readily calculate the whole amount of radiant energy supplied by the sun during any assigned period of time. Thus we find

Energy supplied during summer months,	114,840
„ „ winter „	19,080

Roughly speaking, the sun supplies during the summer months in our latitudes nearly 100,000 units of energy per unit area in excess of what it supplies during the winter months. But of this amount only 1200 units accumulate in the crust in the form of heat. In other words only about 1 per cent. of the energy falling on the surface of the earth is allowed to accumulate in the crust of the earth as heat. The remaining 99 per cent. escapes by radiation and convection or is partly reflected back untransformed into heat. This seems to be quite a reasonable result, and contrasts markedly with the extraordinary result given in the first part of the paper.

The above estimate is necessarily of a rough character. In this country the sunshine which reaches the earth's surface so as to be propagated downwards as a wave of heat is on the average much less than what would be in a clear atmosphere similar to that in which Langley worked. Consequently the overplus of energy supplied in the warmer months of the year is probably over-estimated. Then again there is some doubt as to the surface values of temperature as deduced from the Calton Hill thermometers, for a complete account of which I refer to a paper shortly to be published in the *Transactions* of this Society by Mr Heath. Had I been aware sooner of the fact that Mr Heath was preparing an elaborate discussion of the Calton Hill rock thermometers, I should not have taken the trouble to make an harmonic analysis of the

eight years' observations already published by Piazzi Smyth. These I have used as they were given, without any regard to the probable corrections. As my object was, however, to get an approximate estimate of the amount of heat stored in the rock at different times, and not to discuss the conductivity of the material, it was not necessary to pay much attention to comparatively small errors of observation. The probable heterogeneity of the different layers and the surface irregularities of the rock itself will give rise to disturbances as important as any that might arise from neglect of slight and (as Mr Heath has pointed out) not very certain corrections.

It would be of great interest to apply similar calculations to underground temperatures in other parts of the globe, especially in parts which are blessed with fairly steady sunshine.

In regard to the general form of the curves of underground temperature, there is one feature which I do not remember to have seen commented upon. The feature is apparent in all, but most evident in the curve for the thermometer nearest the surface. It is the sharpness of the crest as compared with the trough. The reason of this is at once recognised when we observe that exactly the same feature is distinctly characteristic of the lower solar radiation curve, but not so of the higher curve. In other words, in the higher latitude the low altitude of the sun and the shortness of the day combine during the winter months to produce a marked effect upon the law of absorption of solar energy. In lower latitudes this effect is hardly appreciable, and at the equator a perfectly symmetrical semi-annual variation of comparatively small amplitude is to be expected. It is instructive to compare the annual variations of solar radiation already given for two different latitudes with the corresponding variation at a place on the equator. The results, obtained in exactly the same way, are as follows:

SOLAR RADIATION AND EARTH TEMPERATURES.

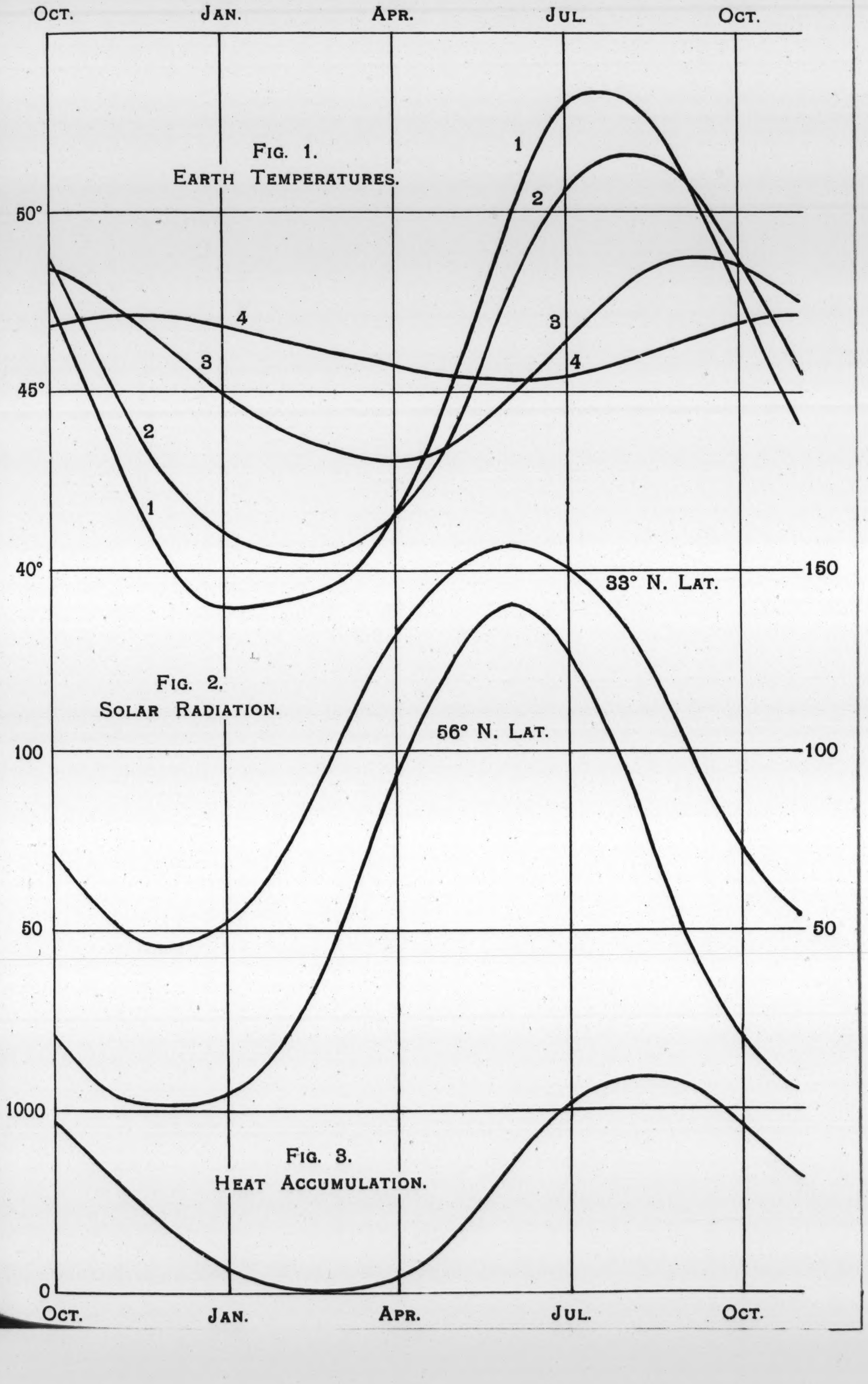


Table showing the time in hours reckoned from culmination at which for given values of the sun's declination, as shown in the top row, the radiation crossing unit horizontal surface at the equator has value as shown in the first column.

R.	23° 27′	20°	12°	0°	Sun's declination.
0·7				0	hours measured from culmination.
·679			0		
·643		0			
·622	0				
·606	0·77	1·12	1·55	1·68	
·512	1·98	2·11	2·34	2·46	
·421	2·69				
·331	3·27	3·35	3·48	3·54	
·249	3·79				
·091	4·73	4·76	4·81	4·83	
·06	4·94				
·007	5·47	5·49	5·51	5·52	

Declination.	Half-daily heating (relative).
+23° 27′	122·9
+20	127·4
+12	135·2
0	139·2
−12	135·2
−20	127·4
−23 27	122·9

Earth Thermometers at the equator would, of course, show no annual period; and the semi-annual period would penetrate to a comparatively small depth.

(Delayed in publication.)

Change of the Coefficient of Absorption of a Gas in a Liquid with Temperature. By Professor Kuenen. (With a Plate.)

(Read January 22, 1900.)

Bunsen's classical experiments on the absorption of gases by liquids show that the coefficient of absorption in water and alcohol between 0° and 20° diminishes as the temperature rises. Bohr and Bock [1] found that at higher temperatures the coefficients of some gases (hydrogen and probably nitrogen) pass through a minimum, hydrogen in water at 60° C., nitrogen not far from 100° C. These results were not confirmed by Winkler,[2] who concluded from his experiments that the coefficient approaches a smallest value asymptotically. Recently Estreicher,[3] working with Professor Ramsay, found a minimum in the solubility for helium in water at 25° C.

By a letter from Professor Ramsay I was induced to look at the problem from the general point of view of mixtures, and to consider whether the phenomenon was not connected with the approach of the critical region.[4]

Hitherto mixtures of water or alcohol with gases have not been investigated up to the critical condition; instead of these liquids, however, we may consider a substance like methyl chloride or carbon dioxide, whose critical temperatures are more easily accessible, and mixtures of which with substances of low critical point have been sufficiently investigated for our purpose. In the vast majority of cases, mixtures of two substances of widely different critical temperatures and vapour-pressures behave in very much the same manner, and from the behaviour of a combination like methyl chloride and carbon dioxide,[5] or carbon

[1] *Wied. Ann.*, 44, p. 318. [2] *Zeitschr. f. Physik. Chemie*, 9, p. 171.
[3] *Ibid.*, 31, p. 176. [4] *Vide* Estreicher, *loc. cit.*, p. 186.
[5] Kuenen, Communications, Leiden, No. 13, *Zeitschr. f. Physik. Chemie*, 4, p. 673.

dioxide and hydrogen,[1] we may with safety draw conclusions with regard to combinations of water and alcohol with a gas.

It may here be mentioned that the thermodynamical theory of mixtures does not lead to a definite law for the variation of the coefficient of absorption with temperature, unless special assumptions are made with regard to the equation of condition of the mixture and the constants which it contains. But even without doing that, our present knowledge of the behaviour of mixtures of the kind indicated above enables us to show the direction in which this coefficient will change.

The coefficient of absorption, as used by Bunsen, represents the volume of the gas, reduced to 0°, which is absorbed by unit-volume of the liquid. This gas-volume is, by Boyle's law, independent of the pressure as long as Henry's law holds, *i.e.*, as long as the quantity of gas absorbed is proportional to the partial pressure of the gas. This law is in many cases sufficiently correct for low pressures, and as long as the temperature is not too high. On the other hand, it cannot be true near the critical point of the liquid; the absorption of the gas lowers the critical temperature of the liquid, or, to put it more correctly, it gives a mixture whose critical point is lower than that of the liquid. The consequence is that if the temperature is near the critical point of the liquid, the absorption itself may make the liquid disappear, and the law of absorption is naturally no longer valid.

It is easily seen how we have to modify the definition of coefficient of absorption so that we may still use it when Henry's law begins to fail. Instead of considering the volume of gas absorbed reduced to 0°, or, which comes to the same, the gas-volume reduced to 0° and normal pressure, divided by the partial pressure of the gas, we must take the limiting ratio of the latter quantities for infinitely small absorption. For the sake of simplicity of expression, we may substitute mass for "volume reduced to 0° and normal pressure." Finally, it will be more convenient as well as more natural to consider the mass of gas absorbed by a constant mass of liquid instead of by a constant volume of liquid, a modification which does not affect appreciably results obtained at low temperatures, but will make itself felt as the liquid

[1] Verschaffelt, Communications, Leiden, No. 45.

begins to expand. We shall thus call coefficient of absorption "*the rate at which the mass of the gas is absorbed by unit mass of the liquid per unit partial pressure*"; by partial pressure is meant the pressure of the liquid mixture diminished by the vapour-pressure of the pure solvent. Up to a small distance from the critical point there is no harm in substituting for "mass absorbed per unit pressure" the ratio of mass absorbed and pressure, if only small pressures are considered.

In considering the value of the coefficient of absorption in a special case, I shall use the vapour-pressure temperature diagram, for a complete discussion of which I must refer to former papers.[1] The figure gives the general appearance of the diagram for two substances of widely different critical temperatures and vapour-pressures, in this case methyl chloride (solvent) and carbon dioxide (gas dissolved). It contains in addition to the two vapour-pressure curves of the pure constituents, ending at C_1 and C_2, the two critical points, condensation-curves for some of the mixtures. Each one of these belongs to a mixture of given composition; the lower branch of the loop gives the pressures and corresponding temperatures at which the mixture in its lighter condition (*i.e.*, as vapour) is in equilibrium with a liquid mixture of different composition, the upper branch contains the points at which the mixture as a liquid is in equilibrium with a vapour.

In our problem we have to deal with the latter, the upper branch: its vertical distance from the vapour-pressure curve of the solvent is what we have called the partial pressure of the gas, and the quotient of the (constant) mass of the gas which the particular mixture say of the lowest loop shown in the diagram contains and this partial pressure is the coefficient of absorption. Obviously then the coefficient of absorption is inversely proportional to the vertical distance of the upper branch of the loop and the methylchloride-curve.

Owing to the peculiar way in which the upper branch of the loop bends round on approaching the critical curve, C_2 P C_1, it will be seen that the partial pressure referred to will necessarily in the end diminish and therefore the coefficient of absorption increase. At low temperatures the partial pressure is low and the

[1] *Phil. Mag.*, 40, p. 175.

coefficient of absorption relatively high, and there must therefore be a minimum somewhere. With strongly soluble gases (for which the condensation curve is a narrow loop) this minimum will probably occur at a relatively high temperature not far from the critical point. For sparingly soluble gases on the other hand we may expect a well-marked minimum at lower temperature. The minimum will therefore occur at low temperature for helium, hydrogen and nitrogen in water, at a higher temperature for oxygen and argon, conclusions which are borne out by the experiments referred to.

It is incorrect to say [2] that the coefficient becomes infinite at the critical point. The partial pressure does not and cannot approach zero, and the coefficient of absorption remains finite. That this assertion is true even if we apply the correct definition which holds up to the critical point may be shown as follows. We may treat the lower branch of the condensation-curve in the same manner as we have treated the upper—*i.e.*, we may consider the partial pressure of the gas in the vapour-mixture and introduce a coefficient of absorption of the gas in the vapour—viz., the ratio of the mass of the gas contained in the vapour-mixture in the saturated condition per unit mass of the solvent and the partial pressure of the gas. If we call the density of the saturated vapour of the solvent d_1, the density of the gas at one atmosphere d, its partial pressure p and the mass mixed with unit mass of vapour m, we have by Dalton's law

$$\frac{1}{d_1}=\frac{m}{dp}$$

or

$$\frac{m}{p}=\frac{d}{d_1}.$$

Approximately, therefore, this new coefficient of absorption is equal to the ratio of d and d_1: as the temperature rises d diminishes as $(1+\alpha t)^{-1}$ and d_1 increases, so that the coefficient is steadily diminishing with increasing rapidity. It is easily seen that this conclusion holds even if we take the limiting ratio of m and p. Owing to the existence of the condensation-loop the coefficient of absorption in the vapour ultimately approaches and

[1] Estreicher, *loc. cit.*, p. 186.

coincides at R with the coefficient of absorption in the liquid, which, as we saw, is on the increase in the critical region. Obviously then the latter does not approach infinity.

The same result would have been arrived at if we had considered the gas absorbed in unit volume instead of in unit mass of the liquid, but we could not in that case have used the diagram which is drawn for mixtures of constant composition.

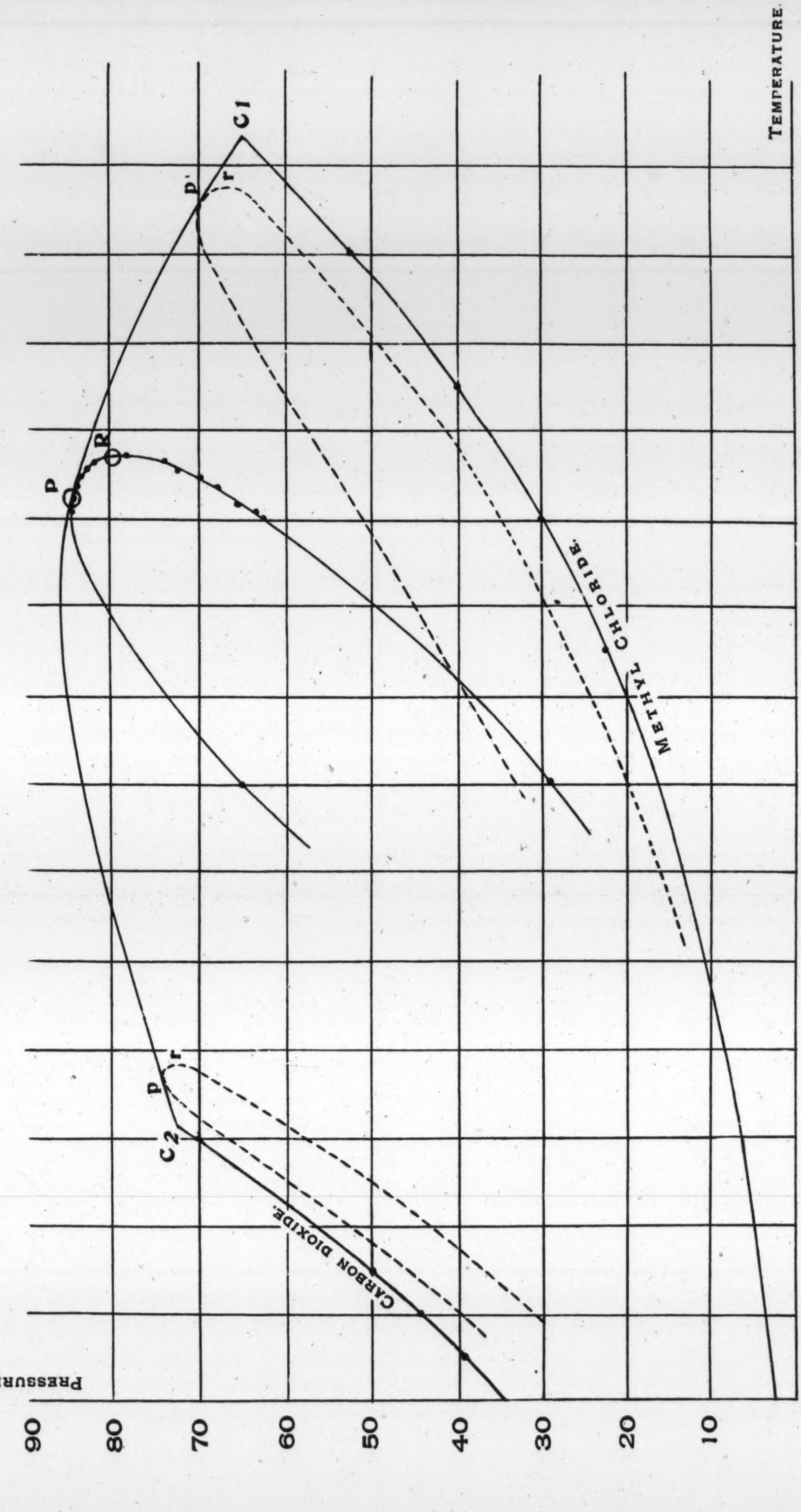
It might be tried to use the coefficient for a mixture in the homogeneous condition—*e.g.*, above the critical point, *i.e.*, to the right of the critical curve; at moderate pressures the approximate formula

$$\frac{m}{p}=\frac{d}{d_1}$$

still holds, but d_1 is not now a constant as it was for saturated vapour but is proportional to the partial pressure of the vapour; by changing the amount of the solvent we may under these circumstances give the coefficient any value we like. In this case it would be better to consider the gas dissolved in unit volume. The formula then becomes

$$\frac{m}{p}=d,$$

which gives an approximately constant value for the coefficient at a given temperature. But in any case no special advantage attaches to the use of the term in this case, and it seems more appropriate to reserve it for conditions of equilibrium between a vapour and a liquid.



CHANGE OF THE CO-EFFICIENT OF ABSORPTION OF A GAS IN A LIQUID, WITH TEMPERATURE.

A. RITCHIE & SON, EDIN^R.

Simple Proof of Gibbs' Phase-rule. By Professor Kuenen.

(Read January 22, 1900.)

About a year ago, while writing a Text-book on Heat in which the use of higher mathematics had to be avoided, it appeared to me that the phase-rule could be rigorously proved by a process which does not involve the deduction of the somewhat difficult thermodynamical equations used by Gibbs, Planck and others. Quite lately, however, I discovered that proofs somewhat similar to mine had been previously given by Nernst and Bancroft, and I must therefore not be understood to claim originality in this paper. Seeing, however, that modern thermodynamics do not yet command in this country the interest they deserve, it will not be superfluous to draw the attention of the Society to the subject.

The phase-rule states that when n mutually independent substances are in equilibrium in a system of r phases, the system is capable of $(n-r+2)$ independent variations, or, the number of independent variable quantities is $(n-r+2)$.

In determining n we must not count separately those substances which in all the phases (either separately or in combination with others in the ratio in which they occur in the same phase) may be formed out of those that have already been counted, with the additional understanding that if we obtain different results for the total number by counting in a different order, we are to take the smallest of the numbers found.

A system of ammonium chloride and its products of dissociation, ammonia and hydrochloric acid, must therefore be considered to contain *one* substance, if the two substances are present in equivalent quantities, *two* substances, if there is a surplus of either of the two gases. Calcium carbonate, on the other hand, when dissociating, contains *two* substances, as neither the carbon dioxide in the gas-phase nor the calcium oxide can be formed out of the calcium carbonate by itself; two independent substances, say carbon dioxide and calcium oxide, are sufficient, as the third substance, carbonate, is formed by the combination of

the other two. A single substance, whose molecules are supposed to associate into groups of two or more, must still be looked upon as one substance from the point of view of the phase-rule.

The condition of each phase is determined by $(n+1)$ quantities, viz., the $(n-1)$ ratios in which the n substances occur in it and two additional quantities, say the temperature and the pressure. As, however, the last two are the same in all the phases, the total number of variables is $(n-1)r+2$. (If there are semi-permeable walls, the pressure is not the same in all the phases, and the phase-rule does not apply in its usual form.)

In order to prove the phase-rule, we have to apply the second law of thermodynamics. For our purpose we may put it in this form, that the system must take up a condition of equilibrium; otherwise we should get a perpetuum-mobile; there must, therefore, be an equation to be satisfied by the variables for every independent virtual reaction in the system.

Apart from the conditions that the temperature and pressure are the same in all the phases which arise from the fact that an irreversible transference of heat or irreversible expansions are excluded, we thus obtain one equation for the virtual transition of every one of the n substances between every combination of two phases. If all these combinations had to be taken separately, we should have $\frac{r(r-1)}{1 \cdot 2} \times n$ equations in all, but from the second law we conclude at once that the equilibrium between one phase and all the others separately involves that between every combination of these last. The total number of equations is therefore $(r-1) \times n$ and the number of independent variables:

$$(n-1)r+2-(r-1)n=n-r+2.$$

Q.E.D.

The Biology of the Genus Pissodes. (George Heriot Research Fellowship Thesis.) By R. Stewart MacDougall, M.A., D.Sc. *Communicated by* Professor Cossar Ewart.

(Read June 4, 1900.)

In the case of any harmful insect of economic importance, in order to war against it, or apply remedial measures at all intelligently, a knowledge of the life-history of the pest is necessary. This proposition will, I think, meet with such ready acceptance as to render proof unnecessary, but I might in illustration mention two cases which came under my own observation, where in the one case a knowledge of the round of life of the attacking insect saved a whole forest, and in the other proved of great importance.

There is a large moth, not uncommon in the pine woods on the Continent, viz., *Gastropachi pini* (Ochsh), whose caterpillars sometimes do enormous damage by stripping the pines of their needles. Some years ago there was a plague of these moths in the extensive Royal Forest near Nürnberg, in Bavaria. The moths had laid their eggs in July on the needles and branches, and the caterpillars which hatched out had fed in tens of thousands on the trees during August and September. They left the trees in October and November to pass the winter in sheltered places under the moss and litter of the forest. As a point in their biology, it was known that in the following March they would come out of their hiding-places and reascend the trees to complete their growth. A ring or circle of very sticky tar was therefore placed round each tree in the month of February. The result was that the caterpillars, endeavouring to ascend the trees after the winter's rest, were brought to a halt at the rings, which they would not cross, and here they were massacred in their thousands, and the forest saved.

In another part of Bavaria, where in 1890–91 the attacks of the caterpillars of the Nun moth (*Liparis monacha*) on spruce cost the Government £100,000, a new point in the biology, which had escaped notice in the previous devastations of this moth, came

to light, and its recognition suggested an excellent annihilative measure against the caterpillars. It was observed that the Nun caterpillars in the beginning of June, and for some weeks thereafter (in the hot weather), had the habit of leaving the trees in the day-time to hide in the moss below, perhaps to escape the heat of the sun, perhaps to avoid their enemies the parasitic Tachinidæ flies. These caterpillars ascended the trees again at night to feed. This was one of the reasons which suggested the use of tar-rings here too. The descending larvæ would not pass the ring, but collected over it, and thus thousands came into the power of those whose work it was to go round and destroy them, which otherwise, without the knowledge of this habit, would never have been reached.

Now, although the Pissodes species have been long known as forest pests, the contradictory accounts given of their generation (and the flight times and length of time taken for development of such tree-infesting forms determine the time for trapping them by means of catch-trees), as well as my own observations of the species, satisfied me that something was still to be discovered. The results of my experiments, especially as these prove a long-continued egg-laying on the part of the mother beetles, with a very long imago life of both sexes, will, I hope, not only prove of interest on their scientific side, but will place on a sure and logical foundation the defensive and offensive methods of procedure against these enemies of our woods.

As the best method of procedure against bark-boring beetles is the employment of decoy stems or catch-trees or bark traps (the details varying with the species), a knowledge of the correct times when these should be prepared and revised and examined is the very kernel of the treatment.

Some of the foremost economic zoologists on the Continent, in their recommendations regarding tree-infesting Coleoptera, attach, it seems to me, too great an importance to what they call the 'spring swarm' or the 'summer swarm' or the 'autumn swarm.' The life-histories are written of as if the egg-laying of a species and resulting issue of the brood of beetles were confined to definite times, limited in extent. Those holding this opinion recommend the preparation of the decoy stems only against these swarm periods. It would be extremely agreeable if we could rely

on such a perfect periodicity, but the opinion, for its truth, takes for granted a comparatively short life in the adult stage, with the eggs all laid about the same time, and a rate of larval feeding extremely regular. But this does not hold even of the Bostrichidæ, which are quoted as a good example of it. Again and again I have taken members of the same species of Bostrichidæ at the same time, and yet in very different stages of development.

It is true that the intervention of winter produces a certain periodicity, inasmuch as the last-appearing beetles of the previous year and the earliest-appearing in the spring will start egg-laying at the same time; but that mature beetles of the same species can issue and proceed to breed in any of the warmer months can no longer be doubted. Outside of the Bostrichidæ, Von Oppen (1) proved this in 1885 for *Hylobius abietis*, the large pine weevil, and now my experiments have proved that for the Pissodes no longer can the preparing of catch-trees be limited to so-called swarm periods, but must be attended to all the year from March till October.

Position of the Pissodes in the Insect World.

Of the families into which the Rhyncophora or proboscis beetles are broken up, one is the Curculionidæ, and to it the Pissodes belong.

The Curculionidæ may be defined as rounded or oval beetles, possessing a beak of varying length, and distinctly elbowed antennæ; the females do not enter bodily into the tree for the purpose of egg-laying like the Scolytidæ, but lay their eggs on the tree externally (rarely), or in a hole bored from the outside (generally), or, it may be, lay them in the soil.

This family contains a very large number of genera, many of which are very important from the economic standpoint. The harm may be done by the grubs, more rarely by the imago, and rarest of all by both.

Among the forms with destructive grubs are Otiorhynchus, whose larvæ, hatching from eggs laid in roots or in the ground in their neighbourhood, gnaw the external surface of these and cause decay; our genus Pissodes; the grub of *Cryptorhynchus lapathi*, so

harmful to the alder; the leaf-mining larvæ of the lively *Orchestes fagi*; the grub of *Balaninus nucum*, familiar in nuts; and the Anthonomus larvæ, so troublesome to the apple grower. Harmful in the mature stage is *Hylobius abietis*, the pine weevil, one of the greatest scourges in our conifer plantations and nurseries.

The Genus Pissodes.

The species belonging to this genus have a longish rostrum. Near the middle of the rostrum the elbowed antennæ are inserted, their long basal joint almost reaching the small, slightly-projecting eyes. The prothorax is narrowed in front, and its posterior margin, on examination with a lens, may show two slight excavations. The scutellum is round and raised. The elytra quite cover the abdomen. Femur untoothed, tibia straight and with a curved hook at the point. The third joint of the tarsus is broad and two-lobed, and the terminal fifth joint ends in two simple claws.

Life History.—In life history most of the Pissodes agree. The females lay their eggs in the bark of conifers. The hatched-out grubs, starting, it may be, from a common centre, gnaw long winding tunnels in the bark, the whole perhaps showing a star-like pattern, although this design is not so frequently met with in *Pissodes notatus* and *Pissodes piniphilus*. The full-fed grubs gnaw in the outermost layers of the wood a little bed or cradle, oval in shape, and here, covered by a cushion of wood chips and sawdust, they pupate, the imago biting its way when ready through bed-cover and bark, leaving a small round hole.

The grubs living and tunnelling between the bark and the wood interfere with the conduction of the sap, and the infested plants weaken and die. While the larval stage is the very injurious one, the adult beetles may weaken the plant by the punctures they make with their probosces when feeding.

Of the twenty or so species known, five are well known in Great Britain or the Continent as pests on coniferous trees—viz., *P. pini*, *P. notatus*, *P. piniphilus*, *P. piceæ*, *P. harcyniæ*.

My experiments have been with the three British species, *P. notatus*, *P. pini*, *P. piniphilus*.

Determination of the Species.

The accompanying table is, with slight modifications (I have added scabricollis), that of Professor Nitsche (2).

Posterior corners of prothorax right-angled or projecting somewhat sharply. The upper surface of the prothorax wrinkled and covered with a number of closely-arranged punctures.	Wing covers with a narrow transverse band behind their middle. *P. pini.*	
	Wing covers with a broad transverse band behind their middle.	Wing covers have longitudinal rows of large dots varying in size. *P. piceæ.*
		Wing covers with longitudinal rows of equally-sized dots. *P. notatus.*
Posterior corners of prothorax rounded and the deep punctures not so close together.	Beetles black. *P. harcyniæ.*	
	Beetles rusty brown. *P. piniphilus.*	
	Beetles with a more or less prominent raised middle line on the prothorax. Generally much smaller than harcyniæ and not so black. *P. scabricollis.*	

In the Continental literature on the Pissodes, another form is mentioned—viz., *P. validirostris,* which was said to breed in pine cones. I have proved, however, that *P. notatus* and *P. validirostris* are one and the same (3).

A glance over the above table will show that the species resemble each other closely. This resemblance is close, in size, and colour, and round of life. Besides, the characteristic spots and bands (these latter formed from the coalescence of individual scales), so helpful in the determination of fresh specimens, get rubbed off in course of time, making the recognition of isolated not-fresh specimens troublesome.

Size and colour of species also fluctuate within limits. For example, while a normal-sized *P. piceæ* is not to be confused with a normal-sized *P. notatus,* I have taken specimens of piceæ as small as an ordinary notatus, and not to be distinguished from the latter save by their different food plant.

In the forest one may meet with plants and trees that have

been attacked, but with no insects remaining to suggest the pest. In such cases, as an aid to determination the following may be helpful :—

(*a*) *The larval tunnels may arise from a common centre.*—There is just the chance of confusing the work with that of the Scolytidæ, but in the case of Pissodes no mother tunnel is found, only larval ones. Sometimes the eggs are laid singly. The resulting single tunnels are difficult to determine, but if they are very long one can pretty safely diagnose them as the work of a Pissodes.

(*b*) *The tunnels are long*, a considerable distance intervening between the place of egg-laying and the pupa bed. Recently I took specimens of *P. pini* with larval tunnels a foot long. If the tunnels, for some reason or other, instead of winding on, form a sort of interlacing network confined to one place, then the work may be confused with the larval borings of some of the longicorn beetles. More than once I have found under the bark Pissodes larvæ and Longicorn larvæ working side by side—*e.g.*, once in an old felled silver fir, where among hundreds of larval piceæ were very many grubs of a Rhagium.

(*c*) The pupal beds with their coverings of sawdust and wood-chips.

The pupa of *Hylobius abietis* also lies in such a bed, but is chiefly confined to stumps and roots; besides, it is larger.

(*d*) Typical host plants :—

P. notatus, on pine and in pine cones.
P. pini, on pine, rarely on spruce.
P. piniphilus, on pine.
P. piceæ, on silver fir.
P. harcyniæ, on spruce.
P. scabricollis, on spruce.

My experiments were conducted with the first three in the above list, all three being found in Great Britain.

Pissodes notatus (F.).

How I got my material.

In the month of June 1895, while engaged in entomological work in Bavaria, through the kindness of Professor Pauly, the

State entomologist, I received a number of young (three and four years old) Scots pines, which had become sickly and had died off from insect attack. On examining these I found the beds of *P. notatus*, and therefore enclosed the pines in a sack in order that I might get the imagos when these emerged later on. I left Munich on July 20, 1895, bringing with me the pines to Edinburgh, and on opening them out on July 23rd I found that a number of beetles had issued. With the material thus won I started the experiments at the Royal Botanic Garden, Edinburgh, in a part of the garden very kindly placed at my disposal by Professor Bayley Balfour, to whom I am also indebted for some of the pines used.

Method of Experiment.

During my work in Munich I had become acquainted with the 'sack-method' practised by Professor Pauly in his insect-breeding experiments. In dealing with bark- or wood-boring insects whose development lasts for some months or longer, it is neither convenient nor always possible to make use of entire stems, and yet if branches or sections of the trunk be kept for use, there is always the drawback of a rapid drying. In a cut piece of stem evaporation takes place chiefly from the cut surfaces, and to reduce this evaporation Pauly recommended the paraffining of the cut ends. Both ends of the cut length of stem are dipped several times in melted paraffin, which dries as a thin protective skin over the cut surfaces. That by this means, in spite of evaporation, moisture is retained long enough for the contained insects to complete their development, Pauly's successful breeding experiments with Bostrichidæ prove. Personally I have also proved its value. The paraffined lengths of stem are placed in a sack made of some thin material, and the insects to be experimented with are placed inside and the sack securely tied.

I employed this method at the beginning of my experiments with notatus in 1895, but soon departed from it, as I saw that by it I could not obtain sure results as to one important part of my inquiry—namely, the length of life of notatus in the imago stage. Besides, I was desirous of giving as natural conditions as possible, and after some thought devised the following plan.

I used young pines from three to five or six years of age. Each pine as it was required was uprooted from the nursery or plantation, and after being subjected to careful scrutiny to make sure it was quite free from insect attack, it was immediately planted in soil in a 'pot' large enough to conveniently hold it. To surround the pines I had sacks made 30 inches high by 60 inches in circumference, or 40 inches by 80 inches, etc., according to the size of the pine. The sacks were open at both ends. Over each potted pine such a sack was slipped. It was securely tied round the top of the 'pot,' and stakes were inserted into the soil of the 'pot,' and on these the folds of the sack rested. A counted number of specimens of beetle was then placed on the pine, and the sack secured at the top.

The material of which the sacks were composed consisted of the very thinnest muslin. So thin was the muslin that the Pissodes could be seen from the outside, crawling up on the inside of the bag. Each potted and muslined pine was then placed outside in the garden, quite exposed to all weathers, and except that the enclosed beetles were protected from outside enemies like ichneumon flies and birds, their condition may be described as natural. To give the pines every chance as regards their health, the pots were sunk in the soil up to their rim.

At certain intervals the sacks were opened for examination, and when the proper times came round the beetles were looked for and carefully counted previous to their being placed on fresh material. This proved a very tedious part of the experiment, as the beetles being small, and resembling in a very perfect way the colour of the bark of the pine, not to say the soil, much time had often to be spent in searching for them. The pines, thus freed of their feeding beetles, were once more placed outside, each with its bag surrounding it. Now and again, by little dissections, one traced the progress of the developing brood, which, as it issued, was caught inside the muslin bag. To ensure perfect accuracy, if after very careful search the number of beetles previously placed inside was not exactly accounted for, dead or alive, the pine was removed from its pot and most carefully examined previous to its being placed in a new pot.

Description of *P. notatus*.

This red-brown beetle varies a good deal in size, from $\frac{3}{16}$ inch (the smallest which issued in the course of the experiments) up to $\frac{3}{8}$ inch (the largest which issued).

The posterior angles of the wrinkled prothorax project sharply, and its hinder edges show two sinuous excavations. Both the upper and under surfaces of the beetle are powdered with white scales. On the upper surface of the prothorax stand four well-marked white points, and a fifth on the scutellum. The elytra have two transverse bands of scales, one in front and one behind their middle. The front one, which is non-continuous at the suture, is yellowish on either side externally, whitish internally. The hinder band has also the same coloration; it is broader externally than internally, and is continuous right across the wing covers.

The larva is a fleshy, somewhat wrinkled, curled, legless grub, with a brown scaly head and strong gnawing jaws.

Very common in Germany and France, notatus is certainly spreading in Britain. Fowler (4) gives as localities Chat Moss, Sunderland (introduced in ships), and the Dee and Moray districts in Scotland. These, I am sure, must be added to. Within the last months I have obtained it from Aberdeen and from Glamorganshire in large numbers. Our native notatus are reinforced by arrivals from other countries in imported timber and in driftwood. I have notes from South Wales of logs washed ashore, which on examination contained notatus in various stages of development. Perhaps to such arrivals Glamorgan owes its notatus, and here the beetle has recently done grievous harm to pine plantations.

Pissodes notatus is injurious both in the imago stage and as larva, but chiefly as the latter. The mature weevil in its feeding pierces the bark with its proboscis, making a number of tiny holes. Some of the young pines used in my experiments with the beetle have been quite riddled from top to bottom by the feeding weevils, just as if some one had with a needle pierced all over the stem and branches. The proboscis pierces through the cambium to the outermost layers of the youngest wood. The

circumference of the wounds widens from outside inwards, the innermost part being the widest, doubtless from the moving about of the proboscis in the feeding region. In healthy pines little bead-like drops of resin issue from the punctures, and when, after more than a year's time, I have peeled the bark from a still living pine which had held feeding but not egg-laying notatus for a month, the old feeding-places in the cambial region were plainly marked out as tiny red-brown patches. The punctures may be dangerous in another way, as forming convenient entrance holes for the spores of injurious fungi.

The larva tunnels in the bark and between the bark and wood, and where the bark may be thin the outermost part of the youngest wood may be also gnawed away.

The favourite breeding places are young pines from three or four to eight years of age, but trees in the pole stage are also frequented. The favourite host plant is the Scots pine (*Pinus sylvestris*), but in Britain I have also obtained notatus from Austrian pine (*Pinus Austriaca*), and Weymouth pine (*Pinus strobus*). There are Continental records of attack on spruce and larch, but this is exceptional.

Whether the beetles attack and breed in healthy trees is a much-vexed question. In the world of timber-infesting beetles we meet with various demands as regards quality of food. Some are dainty, asking for a better quality of material, some are easier to satisfy, while some are not at all particular. Thus I find *Bostrichus typographus* dainty, while *Hylesinus piniperda* will practically put up with anything.

Now in deciding this question for notatus, I have no hesitation in saying that it asks for a certain quality. While in old trees the weakly and sickly will be chosen, the thinned branches of perfectly sound trees and any part of a healthy young plant can be used for breeding. The beetles bred quite willingly in the young plants I offered them, these being always freshly dug from nursery or plantation, and apart from a slight 'checking' that would follow the transplanting, there could be no possible suspicion of their vigour.

The female after copulation lays her eggs in holes in the bark. If pines in the pole stage be chosen, then as several eggs may

be laid near one another, owing to the sufficiency of room at the disposal of the larvæ, the resulting tunnels show a star-like pattern. In young plants, however, the larvæ on hatching tunnel upwards and downwards. A trail of brown bore-dust remains behind to map out the path of the larva. Arrived at the end of its gallery, the larva gnaws out a hole in the outer layers of the wood, and in this hollowed-out bed, protected by a cover of sawdust and wood-chips, the pupation stage is passed. These beds may be made from the upper part of the stem all down to the ground, and also an inch or two below ground. A very favourite place is immediately below the whorl of branches, where, in an infested plant, one is always sure to find several beds clustered together.

How plentiful these beds may be may be gathered from this, that in a piece of Austrian pine taken in October 1897, measuring 6 inches long and 1 inch in diameter, I counted no fewer than fifty-seven beds; another piece of a three-year-old pine held eight beds within a space $1\frac{1}{2}$ inches long and $\frac{1}{2}$ inch in diameter.

Very often during the experiments I found that eggs had been laid and larvæ developed on the thinner branches, sometimes on very thin twigs as well as on the main stem and thicker parts of the branches. The result was that when the larva came to gnaw out its bed in the wood the whole of the tissue in these thin twigs from centre to outside (pith and wood alike) was eaten away, and in its bed in the hollow, bounded all round only by a thin rind, the larva pupated. In such cases the merest pressure on the branch bent it at these hollowed-out places. More than once when examining my pines I bent the twigs by accident, squashing the enclosed larva or pupa. In nature the wind must, I think, not rarely break off the twigs at such places, when the recognition of the broken or blown-down twigs might prove helpful in calling attention to the pests. This use of the thinnest twigs for egg-laying in my experiment would be partly due to the beetles not having enough of egg-laying room in thicker parts.

If one remove the chip-cover from the bed before beetle escape, the white pupa may be seen lying on its dorsal surface with the rostrum arranged along the under surface of the thorax. When the beetles are ready to escape, they bore a circular hole through bed-cover and bark. Just before and after emergence they are

light coloured, but soon they darken into their normal coloration. The beetles, although they can fly well, are somewhat sluggish on the pines. In collecting them, when touched they would often drop to the ground and lie motionless as if dead. Owing to their colour they are difficult to find on the pines, till one by practice gets to know where to look for them. When buds were present the beetles would often lie between the buds, which sometimes, like the stem, showed proboscis punctures.

The Generation.

In the literature, which is entirely foreign, on the generation and flight times of notatus, very opposite opinions have been expressed, and before giving my experimental evidence and showing where the various theories fail, because founded on a wrong notion of the biology, it will be useful to quote representative opinions.

1st. The generation is a double one, two broods of beetles being produced in one calendar year. Professor Henschel, championing this view, writes thus: (5) "Eine in Mai engebracte, vom gennanten Käfer getödtete 12. jährige Schwarzkiefer ergabam 17 Juni die ersten am 25. die letzten Imagines. Zwei weitere, aus derselbem Kultur entnommene, am 26. August eingezwingerte Pflanzen enthielten bereits Puppen und lieferten den ausgebildeten Käfer (im Zimmer) vom 3. bis 10. September. Es lässt sich heraus auf Folgendes schliessen:—

"(*a*) Die Generation bei *P. notatus* kann sein, oder is Vielleicht sogar normal eine doppelte.

"(*b*) Die aus der Zweiten (Sommer) Generation hervorgehenden, zuerst entwickelten Käfer, fliegen (warme Herbstwitterung vorausgesetzt) zum Theil noch in Herbst aus und überwintern in Freien; oder sie verbringen bei minder günstigen Witterungscharakter den Winter in Puppenlager und verlassen dassselbe erst in Frühjahr und zwar sehr zeitig (erste Märzkäfer). In diesem Falle doppelte Generation möglich."

I think it very unlikely, in Britain at any rate, that two broods can be produced in a year, even in the most favourable weather conditions, but, any way, one cannot safely infer it from Henschel's facts. One has no guarantee that the beetles which issued in June

were the result of eggs laid in the same year; indeed, they are likelier to have been beetles from larvæ which overwintered as such. Besides, even if for the sake of argument we admit that the June beetles were from eggs of the same year, Henschel takes for granted that the so-called summer generation is able to proceed at once to reproduction, a fact which has still to be proved.

2nd. The *generation is a single or annual one.*—Ratzeburg, Nitsche, Altum, Pauly, and Perris all favour the one-year generation (while also admitting the additional possibility of three generations in two years), although there is some difference of opinion as to the details, Ratzeburg holding it to be the general rule that the winter is passed in the imago stage, while Perris, writing of his observations in France, stands out for hibernation in the larval stage.

Thus Ratzeburg: (6) "Die Generation ist auch meist nur eine einjährige höchstens dann und wann eine anderthalbige, gewiss nie eine doppelte. Die Käfer im Nachsommer oder Herbst ausschlüpfen, überwintern und sich im Frühjahr begatten, so dass man die Brutt im Laufe des Sommers sich vollständig bis zum Käfer entwickeln sieht."

And Perris: (7) "Ordinairement le *P. notatus* hiverne a l'état de larve. Celle-ci se transforme en nymphe vers la fin du mois d'avril ou dans le mois de mai et comme l'état de nymphe dure environs un mois et qu'il faut ensuite a l'insecte parfait un certain temps pour fortifier ses organes, durcir son enveloppe pratiquer une ouverture dans la couche de fibre ligneuses qui formait sa niche et percer enfin le bois on l'écorce qui l'abritait il en résulte que les Pissodes ne se montrent guère que vers la fin de Juin."

The seeming contradictions are really no contradictions at all. The facts are correct, but the generalisation is wrong.

The key to the whole position lies in the proof, given by the experiments, of the long life and long-continued egg-laying of the mother beetles which make it possible to find notatus, at the same time, in very different stages of development. During my experiments I have found with Henschel, imagos in June and August; with Ratzeburg, larvæ in summer and hibernating imagos; with Altum, imagos in May and August; with Perris, hibernating larvæ, and imagos in June and July.

On one and the same day and near one another it is possible to

find eggs, young larvæ, full-grown larvæ, pupæ, and imagos; and the danger of generalising in absence of a complete experiment is further emphasised when I state that I had feeding side by side in the autumn, representatives of three generations of imagos in direct descent, born in 1895, 1896, and 1897 respectively, and among these feeding imagos could be numbered beetles which had issued from my various pines in every month of a year except January, February, March, and December.

Here is a table showing the times when eggs were laid in the course of the experiments.

Tables of Times of Egg-laying.

Year.	No. of Pine.	Length of time Notatus was allowed to remain on Pine.	Proof that Eggs were laid.
1896	1	End of March and beginning of April	A new brood issued.
,,	2	April 17 onwards	,, ,,
1897	12	April 15–May 10	,, ,,
,,	14	April 21–May 29	Got larvæ on dissection.
,,	15	May 1–May 29	A new brood issued.
,,	16	May 10–May 25	,, ,,
,,	17	May 25–June 3	,, ,,
,,	19	May 29–June 30	,, ,,
,,	20	June 3–June 29	,, ,,
,,	27	June 29–July 10	Larvæ got on dissection.
,,	29	June 30–July 28	,, ,,
,,	31	July 10–July 28	,, ,,
,,	32	July 12–August 2	,, ,,
,,	35	July 17–July 31	,, ,,
,,	36	July 28–August 9	,, ,,
,,	37	July 31–August 14	,, ,,
,,	39	August 2–August 16	,, ,,
,,	40	August 9–August 27	,, ,,
,,	41	August 14–August 28	,, ,,
,,	45	August 27–September 29	,, ,,
,,	46	August 28–October 1	,, ,,
1898	55	March 14–April 20	A new brood issued.
,,	56	March 23–April 22	,, ,,
,,	57	April 9–May 28	Larvæ got on dissection.
,,	58	April 20–May 10	A new brood issued.
,,	61	May 10–May 27	Larvæ got on dissection.
,,	62	May 27–June 22	Pupæ ,, ,,
,,	63	May 27–June 29	A new brood issued.
,,	64	June 22–July 11	Larvæ got on dissection.
,,	65	June 29–July 21	,, ,,
,,	66	July 11–August 29	,, ,,
,,	67	July 26–August 31	,, ,,

The months of the year in which new imagos have issued from their beds after pupation will be seen from the next table.

Table of Escape Months of Pissodes notatus under natural conditions, as recorded in the series of Experiments.

Year.	Month.	Remarks.
1896	Last week of July	From eggs laid in same year.
„	August	„ „ „
„	September	„ „ „
„	October	„ „ „
„	November	„ „ „
1897	April	These beetles were from eggs laid in 1896. They had reached the imago stage before the entry of winter 1896–97, but they remained in their beds till April and May.
„	May	
„	June	From eggs laid in 1896. Winter passed in beds as full-fed larvæ.
„	July	From eggs laid in 1896.
„	August	From eggs laid in 1897.
„	September	„ „ „
„	October	„ „ „
„	November	„ „ „
1898	April	„ „ „
„	July	„ „ „
„	August	„ „ „
„	September	From eggs laid in 1898.
„	October	„ „ „

I also found, towards the end of March both in 1896 and 1897, beetles feeding on my pines. These were beetles from among those which had issued in the previous summers or autumns, and had early come out of their winter quarters to feed again. Save December, January and February, there is no month of the year in which I have not found feeding beetles. No longer then can the preparation of catch-trees or decoy stems be limited to so-called swarm periods, but must be attended to from March onwards throughout the year.

While in view of this egg-laying from April to September, and the consequent succession of imagos (a succession which, save for

the intervening winter, might be expected to be a perfectly regular one), the old dispute as to the generation loses some of its significance; it is nevertheless of importance to know how long individual development takes.

What, then, is the period of time represented from the egg-laying through the larval and pupal stages and up to the issue of the individual imago?

I give in tabular form some of the results.

Length of Time for Development.

No. of Pine.	Beetles placed on Pine.	First Imagos Issue.	Length of Time.
1	End of March 1896	July 24, 1896	114 to 120 days.
2	April 17, 1896	Aug. 24, 1896	128 days.
3	June 17, 1896	Oct. 15, 1896	119 „
12	April 15, 1897	Aug. 31, 1897	137 „
14	April 21, 1897	Sept. 8, 1897	139 „
16	May 10, 1897	Sept. 24, 1897	136 „
17	May 25, 1897	Sept. 29, 1897	127 „
19	May 29, 1897	Sept. 18, 1897	111 „
20	June 3, 1897	Sept. 20, 1897	108 „

In each case the time is reckoned from the day on which the beetles were placed on the plant.

To take the general results given in the table, without comparison of different weather conditions, the shortest period taken for development was three and a half months, and the longest, four and a half months, showing an average over nine cases, extending from April to June, of four months.

Very different, however, is the result if the larva be overtaken by the winter, the period of the development extending then over ten or eleven months, *e.g.*, pine 3 held in November 1896 full-grown larvæ in their beds, and these did not reach the imago stage till—the earliest on June 24th, and the last on June 27th, 1897, over ten months since this pine had been left free from beetles. This is further shown in the accompanying table:—

No. of Pine.	Length of time Notatus on Pine.	Date of Issue of first of New Brood.	Length of Time.
31	July 10 to July 28, 1897	Apr. 28, 1898	Over 9 months.
37	July 31 to Aug. 19, ,,	July 20, ,,	About 12 months.
39	Aug. 2 to Aug. 16, ,,	,, ,, ,,	,, ,, ,,
40	Aug. 9 to Aug. 27, ,,	,, 25, ,,	,, ,, ,,

It is impossible, however, to lay down a hard and fast rule as to length of time for development from egg to imago, for I have found and been surprised at the great variation shown in rate of growth and imago escape where eggs had been laid by the same beetles, on the same plant, and within a comparatively short interval of time one from the other. The part of the plant the eggs are laid in; the difference in quality of food in different parts of the same host plant, so that some larvæ will feed in better places and others in worse; the possibilities of overcrowding from much egg-laying so that feeding larvæ will interfere with one another; all these influence development in one direction or the other.

In illustration of the foregoing, and especially to show that issue of adult beetles from a pine may last over a much longer interval of time than that represented between the laying of the first and the last egg, I subjoin details of imago issue from some of the experimental pines.

Pine 2.

This pine (a four-year-old one) held in it from 17th April 1896 till about the middle of June 1896, 16 notatus. The first new imago appeared on 24th August 1896 and the last on 7th July 1897.

On 2nd April 1897 I uprooted this pine, which had been standing all winter exposed to the weather. On being examined, the part of the pine immediately under the surface of the soil, for a depth of 2 inches, showed a number of little round exit holes from which adult beetles peeped out. One of them on being touched walked

Date of Issue of New Imagos.	No.	Date of Issue of New Imagos.	No.
August 24, 1896	1	September 17, 1896	2
„ 27, „	1	„ 18, „	1
„ 28, „	1	„ 19, „	2
„ 31, „	1	„ 20, „	4
September 1, 1896	1	„ 22, „	2
„ 4, „	2	„ 24, „	3
„ 5, „	3	„ 26, „	2
„ 7, „	1	„ 27, „	1
„ 8, „	2	October 5, 1896	1
„ 10, „	1	„ 12, „	1
„ 11, „	2	November 2, 1896	2
„ 12, „	2	„ 13, „	1
„ 14, „	3	„ 23, „	1
„ 15, „	1		

out, but some were dead. I did not replace this pine in the soil, but kept it in a muslin bag. On 8th May 1897 another notatus issued, on 4th July four more, and the last two on 7th July 1897.

Pine 12.

This pine held 4 notatus, 2 male and 2 females, from 15th April to 10th May 1897.

Date of Issue of New Imagos.	No.	Date of Issue of New Imagos.	No.
August 23, 1897	3	September 11, 1897	1
„ 24, „	1	„ 12, „	2
„ 25, „	1	„ 13, „	1
„ 26, „	3	„ 14, „	1
„ 27, „	3	„ 24, „	1
„ 29, „	1	„ 25, „	1
„ 30, „	1	October 4, 1897	1
„ 31, „	1	„ 23, „	1
September 1, 1897	1	„ 24, „	1
„ 2, „	1	November 1, 1897	1
„ 8, „	1	„ 8, „	1
„ 9, „	1		

On 24th December I uprooted this pine, and on dissection over the whole plant I found other nine beds with pupæ or larvæ in them. These nine beds were all on a part of the pine below ground.

Pine 14.

This pine held 12 notatus from 21st April 1897 till 29th May 1897.

Date of Issue of New Imagos.	No.	Date of Issue of New Imagos.	No.
September 8, 1897	1	September 30, 1897	2
„ 15, „	1	October 6, 1897	1
„ 18, „	1	„ 7, „	1
„ 23, „	2	„ 9, „	1
„ 25, „	1	„ 16, „	1
„ 27, „	2	„ 17, „	1
„ 28, „	1	„ 19, „	2
„ 29, „	2		

On 24th December 1897 I uprooted this pine, and on dissection found an inch or two below ground several beds. Of these beds four were touching one another, one of them held a perfect beetle, one a pupa, and the other two full-grown larvæ.

Pines 17 and 18.

These two pines were not very healthy. I placed them both in one large pot on 25th May 1897. On this date four notatus, two male and two female, were placed inside, and removed on 3rd June 1897. They were thus on the pine only 9 days.

Date of Issue of New Imagos.	No.
September 29, 1897	1
October 5, 1897	1

I uprooted this pine on 31st December 1897, and carefully dissected all the bark away from the pine which so far had given up no beetles. I found near the top of the stem a mature beetle in its bed. Lower down the stem I found feeding larvæ, *i.e.*, larvæ which had not yet begun to make a bed, some larger, some smaller.

In the other pine from which the two beetles had issued, I found on dissection three pupating larvæ in beds below a whorl, while below ground I got larvæ which had not begun to make their beds.

PINE 19.

This was a vigorous young pine which held 22 notatus from 29th May 1897 till 30th June 1897.

Date of Issue of New Imagos.	No.	Date of Issue of New Imagos.	No.
September 18, 1897	2	October 18, 1897	2
,, 22, ,,	2	,, 19, ,,	1
,, 26, ,,	2	,, 21, ,,	4
,, 27, ,,	1	,, 23, ,,	1
,, 28, ,,	2	,, 26, ,,	2
,, 29, ,,	2	,, 28, ,,	1
,, 30, ,,	2	,, 31, ,,	1
October 1, 1897	1	November 4, 1897	2
,, 2, ,,	2	,, 8, ,,	2
,, 4, ,,	1	,, 9, ,,	1
,, 6, ,,	4	,, 15, ,,	1
,, 7, ,,	1	,, 19, ,,	2
,, 8, ,,	1	,, 22, ,,	2
,, 9, ,,	2	,, 27, ,,	1
,, 10 and 11, ,,	2	April 14, 1898	1
,, 12, ,,	1	,, 20, ,,	1
,, 16, ,,	1	,, 28, ,,	1
,, 17, ,,	1	,, 30, ,,	1

PINE 20.

This healthy and vigorous pine held four notatus from 3rd June 1897 till 29th June 1897.

Date of Issue of New Imagos.	No.	Date of Issue of New Imagos.	No.
September 20, 1897	1	October 13, 1897	1
" 21, "	1	" 15, "	1
" 22, "	1	" 16, "	2
" 23, "	1	" 17, "	3
" 24, "	3	" 18, "	1
" 25, "	2	" 19, "	2
" 26, "	3	" 21, "	2
" 27, "	1	" 24, "	1
" 28, "	6	" 25, "	1
" 29, "	4	" 26, "	1
" 30, "	1	" 28, "	1
October 1, 1897	2	" 30, "	1
" 3 and 4, "	3	" 31, "	3
" 5, "	2	November 3, 1897	1
" 6, "	2	" 6, "	1
" 9, "	1	" 17, "	1
" 10 and 11, "	2	" 20, "	1
" 12, "	2	" 24, "	1

On 24th December 1897 I removed the soil from the part of the pine a little below the ground, and on dissection came on two beds side by side, one containing a pupa and the other a larva.

Before passing away from this part of the subject I would like to refer again briefly to the question of the generation. Limiting ourselves to one cycle, and to the earliest laid eggs of that cycle, let us ask—What is the generation of *P. notatus*?

We have seen that the imagos which issued in July 1895 from the pines brought from Munich fed till the autumn and hibernated on the approach of winter, and how, after hibernation, they copulated in spring 1896, the earliest of the resulting brood appearing in July 1896. These July 1896 beetles wintered in 1896–97, appeared again in spring 1897, and from their copulation then a new brood began to issue in August 1897.

Thus we have an annual generation, one brood in a calendar year. But it may be objected to this that the imagos which issue in early autumn will in the same year of their issue proceed to reproduction and egg-laying, from which eggs beetles would be developed say in June of the next year (winter having been passed

in the larval condition), that is, in time to lay eggs in their turn from which another brood would be developed and issue in September or October of the same year. We would thus have three generations in two years.

I reply to this that I have no proof that the newly escaped beetles of autumn are able to proceed at once to reproduction. They seem rather to require some time for ripening, so that reproduction is delayed till after hibernation.

In 1896 I placed the earliest new imagos of the year on a large pine. The first imago was placed on the pine on 24th July and others added as they issued. On 2nd September I removed the notatus from the pine (there were no fewer than 27 beetles on the pine at the date of removal). This pine on careful examination showed no trace of egg-laying.

Again in the next year on 24th August I prepared a young pine and placed on it the first issuing beetles. Four beetles were placed on the pine on 24th August, and by 31st August there were thirteen notatus on the pine. Other seven were added between 31st August and 11th September. These twenty notatus were allowed to remain on the pine till 7th October. On 27th December I carefully dissected this pine from top to bottom, peeling off all the bark, and found no trace of egg-laying. Still again, between 12th September and 24th September I placed twenty newly-issued notatus on a fresh pine and allowed them to remain till they went into winter quarters in November. Dissection of this pine showed much trace of the feeding of the beetles but none of egg-laying.

It must not be forgotten here that during the August and September I had egg-laying and larval feeding in other pines which during these months held old beetles which had issued in the preceding year.

But while the beetles that issue in late summer or autumn seem not immediately ripe for reproduction, these individuals which have not completed their development because of the entry of winter, but have lain in their beds all winter, are, when they issue in the next year as imagos, able to proceed to an efficient copulation. Doubtless ripening of the reproductive organs proceeds during the long period of rest. Here is the proof. At the end of June 1897 and the beginning of July 1897 there issued from two of my

pines imagos which had passed the winter of 1896–97 in beds as full-grown larvæ, perhaps some as pupæ. I placed nine of these on a pine on 12th July, and removed them to fresh material on 2nd August. The proof that they had bred was afforded on 24th December, when I dissected the plant and found larvæ (I may add that on other material they continued to lay till September), a brood of new beetles issuing in July 1898.

If we start a cycle at this stage we might get three generations in the two years, thus:—Eggs laid in July would give imagos in the following June or July, and these proceeding to reproduction, a new brood might issue in late autumn of the same year, which, overwintering as imagos, would lay their eggs in the following spring, from which imagos would be developed in summer. Professor Nüsslin (8) of Karlsruhe by dissection showed that when beetles issued their genital organs were not fully developed. He believes that Pissodes, which appear in the spring from larvæ, which have overwintered as such, are able sooner to proceed to reproduction than those Pissodes imagos which issue in summer as a result of eggs laid in the same year, these latter imagos appearing with their reproductive organs in a less complete condition, and so a longer time elapsing before they can pair efficiently. Professor Nüsslin also showed this most interesting fact that a female isolated in spring after copulation was able to continue the laying of fertilised eggs all during the summer, and even in the winter still held live spermatozoa.

Length of Life of Imago.

Earlier in this communication I spoke of eggs being laid from April till September inclusive. I wish now to emphasise the fact that the same individual mother beetles which start to lay in the spring live all the summer, and can be found in September still laying. The males also may live through this period, copulating and recopulating. Nor does death of the individual necessarily take place at the end of one such copulating or egg-laying season, but as the cold weather approaches these beetles may go into hibernation, and reappear in the succeeding spring to renew their

copulation and egg-laying. They can live to the close of a second year, and even then need not die.

That such statements, in view of the general impression among zoologists of the shortness of imaginal life (especially of a male that has copulated), will require for their general acceptance careful and undoubted proof I readily admit, and such proof I now proceed to give in detail.

It will be remembered how a number of notatus issued in the end of July and the beginning of August 1895 from pines brought by me from Munich. These notatus fed on material furnished to them till November 1895, when they stopped feeding and went into winter quarters a little below the surface of the soil of the pots.

Towards the end of March 1896 I found on examination that the notatus had come out from their winter's rest and were crawling on the muslin-enclosed plants. Some of these on Pine 1 I noticed in copula on 2nd April 1896. This pine was bred in, and before the issue of the new brood I removed the parent beetles.

Some of the other notatus which had wintered in 1895–96 I placed on Pine 2, on 17th April 1896. This was a day of bright sunshine, and the notatus were seen to copulate riotously. On 17th June I removed the notatus from this pine and placed some of them, along with others of the old beetles from Pine 1, on a new pine—viz., Pine 3. I got a new brood of beetles from Pine 2 in August 1896.

Pine 3 altogether received sixteen old (1895) notatus. In July, when examining Pine 3, I chanced to see two pairs of beetles in copula. These I kept out, and placed them on a small pine by themselves. All these beetles of Pine 3 (including the four I took out and isolated) were now a year old.

During August 1896 my time was so taken up with a Summer Vacation Lecture Course that I had little opportunity to attend to Pine 3. Up to the middle of August, however, I had noticed living notatus on the pine (which was now in poor condition), but when I came to revise my pine at the end of the month, I found the pine dry and dead, and the notatus also dead. This was a disappointment to me, as I might well suspect that the death of the twelve months' old imagos had been due to their lack of proper

food material, the pine being hard and dead. I was still left, however, with the four notatus previously isolated. On 2nd October 1896 I placed these four on a fresh pine, surrounded as usual with muslin, and having thrown a handful of moss on the surface soil of the pot, I placed the pine outside in the garden, giving the protection of a glass roof in case any heavy snowfall during the winter might bring the experiment to an untimely end.

On 5th March 1897, on examining the pine, and pulling aside the moss, I noticed a slight movement of the soil on the surface, and soon had the pleasure of seeing, from the place of movement, one of my old notatus appearing after hibernation.

I replaced the moss, and once more surrounding the pine with muslin, left it outside.

On 20th March 1897, a sunny day, I examined the pine again, and found all the four notatus on the pine. Up to this time the beetles were twenty months old (as imagos), and had hibernated twice, in the winters of 1895–96 and 1896–97.

On 20th March 1897 I removed the four beetles to a new pine, and changed them later on the following dates and with the following results :—

No. of Notatus on Pine.	No. of Pine.	How long on Pine.	Proof of Egg-laying.	Remarks.
4	9	Mar. 20–Apr. 15, 1897	Found larvæ	Dissected this pine before issue of new brood.
4	12	Apr. 15–May 10, ,,	Got new brood of beetles	First new imago issued August 23, 1897.
4	16	May 10–May 25, ,,	,, ,,	First new imago issued August 24, 1897.
4	17 and 18	May 25–June 3, ,,	,, ,,	First new imago issued September 29, 1897.
4	20	June 3–June 29, ,, When removing the beetles on June 29, three of the four were active. The fourth was lying on the soil, and died in a short time. This was a male.	,, ,,	First new imago issued September 22, 1897.
3	27	June 29–July 10, 1897	Found larvæ and pupæ on dissection	
3	31	July 10–July 28, ,,	,, ,,	
3	36	July 28–Aug. 9, ,,	Found larvæ on dissection	

On 1st August, while examining Pine 36, I found one notatus lying dead on the surface of the earth. On 9th August another died. Only one veteran notatus now remained alive.

The record so far is:

1 male died June 29, 1897, aged 23 months.
1 female died August 1, 1897, aged 24 months.
1 " " 9, " "

But the interest does not end here. Apart from the long life of these beetles, it is additionally interesting that a male should be the survivor. I had no doubt of the survivor being a male,[1] but to put the matter beyond doubt, I put this survivor under a bell jar with a female from another pine, and soon the two were in copula.

This surviving male I placed on a new pine on 9th August 1897 with a female of another brood. The further record is:

No. of Notatus on Pine.	No. of Pine.	How long on Pine.	Remarks.
2	40	Aug. 9–Aug. 27	New brood in July 1898.
2	45	Aug. 27–Sept. 29	New brood in 1898.

On 29th September I placed these two beetles on a fresh pine, and up till 28th October I frequently saw them both on the pine, the male being now twenty-seven months old. In the beginning of November I looked again for the two beetles, but could only find the female. In spite of long search there was no trace of the male, and my hope was that it had proceeded to hibernation, nor was the hope vain, for on examination of the pine on 12th March 1898, I found the two beetles feeding on the plant, their probosces sunk deep in the bark. I placed these two beetles on a fresh pine on 14th March, and continued to give them fresh material until the experiment ended by loss of the veteran male. The further record is as follows:—

[1] Owing to the close resemblance of males and females, in order to make recognition of sex sure I had adopted the following plan. When I found two beetles in copula I mutilated them by cutting off the tarsus of a leg, on the right side in the case of a male, on the left side in the case of a female.

No. of Pine.	No. of Beetles.	Length of time on Pine.	Proof of Egg-laying.
52	2	Mar. 14–Apr. 20, 1898.	Got new brood later in same year.
58	2	Apr. 20–May 10, ,,	,, ,, ,,
60	2	May 10–May 27, ,,	
62	2	May 27–June 22, ,,	Larvæ and pupæ got on dissection.
64	2	June 22–July 9, ,,	Larvæ on dissection.
66	2	July 11–Aug. 3, ,,	,, ,,

On 3rd August 1898 I placed the male on a fresh pine, and observed it alive several times during the month. On 31st August I undid the muslin sack, but could not find the beetle. Further prolonged search for it was also unsuccessful. At the time of this loss the male had lived with me as imago for over three years.

The long life of the adult beetles can be shown in another way. In one series of experiments during 1897 I began on 1st April with thirty-six notatus. These thirty-six were without exception adults which had issued from the experimental pines in August, September or October 1896, and had passed the winter of 1896–97 hibernating in the soil.

From 1st April 1897 till 1st October 1897 these notatus, which were distributed over various pines, were looked for and changed to fresh material at intervals of a fortnight and over. At the last change, in the beginning of October, twenty-seven of these beetles were alive, a fairly equal mixture of males and females. Eggs had been laid from the end of April onwards up to and including September. The further fate of these notatus is as follows:—

Fate of Thirty-six Beetles with which Experiment was started on 1*st April* 1897.

Alive and Feeding on Pines during October and November 1897, previous to Hibernation.	Found Dead during the Year.	Not Found in spite of Search when Removing to Fresh Material.	Lost or Escaped while Changing.	Accidentally Killed.
27	1 on June 21	1 on July 16	1 on July 16	1 on July 31
	1 on July 31	1 on July 31	1 on August 28	
		1 on October 1	1 on October 1	

These hibernating beetles were in November thirteen to sixteen months old, as imagos.

The further fate of eleven with which I continued to experiment on their reappearance in 1898 after hibernation is:

No. of Pine.	No. of Beetles.	How long on Pine.	Proof of Egg-laying.	Remarks.
56	11	Mar. 23–Apr. 20, 1898	New brood issued	Lost one on April 20.
59	10	Apr. 20–May 10, ,,	,, ,,	
61	10	May 10–May 27, ,,	Larvæ on dissection	Lost one.
63	9	May 27–June 29, ,,	New brood issued	
65	9	June 29–July 21, ,,	Larvæ on dissection	Before examination of this pine on July 21, it had died. Of the nine beetles only four were alive, two of which were males.
67	4	July 26–Aug. 31, ,,	Larvæ on dissection	

In August a gale of wind tore to shreds the muslin surrounding the pine, so that on 31st August 1896 only one notatus could be found. Up to this time these beetles varied in age from twenty-two to twenty-five months, and during this period they had twice hibernated.

Pissodes piniphilus (Hbst.).

Pissodes piniphilus, the pine pole weevil, measures less than a quarter of an inch in size, and in colour is rusty brown, powdered all over with whitish scales. The posterior corners of the prothorax are rounded, being more round than in any other of the Pissodes species: Scutellum, whitish. In place of the ordinary transverse band behind the middle of the elytra there are two large rusty-yellow spots, one on each side, between the suture and the outside edge. These spots are very characteristic, and, along with the absence of the band at the front of the elytra (characteristic of the other Pissodes), are of great service in determination.

Distribution.—This beetle is widely spread over Europe, from France in the south to Sweden in the north. It is said by Fowler to be rare in Britain. Mention is made of it as found at Sunderland in imported timber, and doubtless in this way it has or will spread.

Life history.—This troublesome and sometimes very harmful pest attacks, as its name indicates, chiefly pine forest in the 'pole' stage. While trees from twenty to forty years old are the favourite breeding places, yet piniphilus not seldom attacks old pines, its tunnels being found not in the thick-barked under parts but in the thin-barked upper parts of the branches of the crown.

While larval tunnels of a star-shaped pattern are not unknown, the female piniphilus seems most usually to lay her eggs singly and not several all very close together. On peeling the bark from an attacked stem the larval tunnel is easily traced by the brown-black bore dust which fills it. The tunnels measure from 4 to 6 inches in length, but as each tunnel winds in and traverses the bark at different levels, one is apt to think from the comparatively small part presented at any one level that the tunnels are shorter. The pupal beds gnawed in the wood are small, in keeping with the small-sized weevil, but I find they may go deep; indeed, it would be possible to bark a stem and, yet, owing to the depths of some of the beds in the wood, the enclosed larva or pupa might safely perfect its development. Whilst weakly trees may be preferred, piniphilus also attacks healthy trees. As it makes its onsets high up on a tree, and not on lower more easily seen and examined parts, the determination of attack is rendered difficult.

The forester, up till now, was said to have this in his favour, that piniphilus did not pass through its round of life rapidly, but that as it took two years from the time of egg-laying till the beetles were mature and ready to issue, time was given for observation and procedure against the pest. That this two-yearly generation is erroneous my experiments will show.

The imagos were said to issue in June and the beginning of July, the eggs to be laid in July, and the larvæ to live as such for over twenty months.

Professor Altum (9) founded the theory of a two-yearly generation on the fact that he obtained a brood of piniphilus in 1878 from a dead pine whose spring shoots of 1876 showed normal development while those of 1897 were stunted. He argued from this that if the generation had been a yearly one, as the beetles issued in 1878, the eggs from which they were developed must have been laid, say in June 1877, too late for the resulting larvæ to have

affected the development of the spring shoots of 1877, which in the dead pine would thus not have been found stunted.

The other suggested proof of the development from egg to imago lasting over two years is the finding at the same time of piniphilus, near one another and in very different stages of development. Thus to quote Professor Nitsche (10): "Oberförster Petersen zur Flugzeit 1876 im Walde alle Stadien des Insektes von kaum sichtbaren Larven bis zu flugreifen Käfern. Ebenso fand Nitsche mitte October 1887 in denselbem Rollen zwei ganz verschieden grosse Larvenformen, welche durch keine Uebergänge verbunden waren, also wohl von zwei verschiedenen Jahrgängen herrührten."

I can parallel both of these quoted cases in my experiments, and I will show that this cannot be accepted as proof of a two-yearly generation, but is explained by the fact that like notatus, piniphilus has a long imago life, with an egg-laying which lasts over a number of months. The two-yearly generation of *P. piniphilus*, in view of the smaller size of this beetle compared with other Pissodes species, often seemed to me hard to believe, and this partly suggested the experiment.

At the end of April 1896, through the kindness of Professor Pauly, I had sent on to me in Edinburgh some pine logs, which, on dissection, showed the larval stage of a Pissodes.

After keeping the logs for a short time in water, I placed them in a sack.

On 7th July 1896 the first beetles issued, and on examination they proved to be piniphilus. Escape of adults from the logs continued to 25th July.

Experiments.

Pine log 1.

The first ten piniphilus which issued I placed in a muslin sack with a cut length of sickly pine. The piece of pine was paraffined at both ends, and was allowed to stand in a room with no fire. The ten piniphilus were all dead by 2nd August 1896. After some time I dissected the log, but could find no trace of egg-laying.

Pine 1.

On 13th July 1896 I surrounded a healthy seven-year-old young *Pinus sylvestris*, which was potted, with muslin as in the notatus experiments, and seventeen piniphilus having been introduced, the pine was placed outside. On examination of this pine on 8th October the piniphilus were found alive, and were removed. In the summer of 1897 I dissected this pine from top to bottom. The pine was still alive and healthy, and had made some growth during 1897 in spite of its having been surrounded all the time with a muslin bag. Here and there over the pine were the proboscis punctures made in the previous year by the feeding piniphilus, and on the bark being stripped the brown discoloured spots here and there on the alburnum attested the feeding. There was no trace, however, of eggs having been laid.

Pine log 2.

On 14th July 1896 another pine log was paraffined and placed in a sack. Between 14th July and 25th July twelve piniphilus were introduced, and allowed to remain till 3rd October. Here, again, I could find no trace of any egg-laying.

Pine 2.

On 12th October 1896 I surrounded another potted pine with a muslin covering and introduced thirteen piniphilus, all of them from the brood obtained in July. As this pine was larger than those usually employed, and the muslin sack presented too great a surface to safely allow the pine to be exposed to a high wind, the pot was sunk in the soil in a little glass-house at the Royal Botanic Garden. The door of this house was always left open, and except for the protection of the surrounding glass, which was broken in many places, the weather conditions were the same as outside. One can safely believe that no eggs were laid in October or before the next year. In the soil of the pot and under the moss provided for the purpose these piniphilus hibernated during the winter of 1896–97. On looking over the pine on 2nd April 1897 I noticed

some of the beetles feeding on the plant, showing that for some of them at least hibernation was over. This pine, which was alive but not flourishing, was watered at intervals. On 21st June 1897 the living piniphilus were removed to fresh material. In the month of September the first beetles of the new brood issued, the flight holes being in the upper thinnest part of the main stem. On 1st October another piniphilus issued, and still another on 20th October. On 29th December 1897 dissection revealed a number of beds containing full-fed larvæ. The theory of the two-yearly generation is thus disproved.

As Pine 2 had never been very healthy, at intervals from April onwards I had placed in beside it cut lengths of *Pinus sylvestris*, paraffined at the cut ends so as to give the piniphilus a choice of other and thicker breeding material.

The record from these pine logs is :—

Log.	Description.	How long beside Living Piniphilus.	Proof of Egg-laying.
A	20 inches long and 3 inches in diameter	April 2 to May 5, 1897	On December 29, 1897, on stripping the bark from the log, fifteen larvæ were got. Seven of them lay in beds deep in the wood, three in beds less deep, and two seemed only to have begun to gnaw out their bed. The remaining two larvæ were smaller, and had not reached the full-fed condition.
B	24½ inches long and 2½ inches in diameter	May 5 to June 5, 1897	Dissected on December 29, 1897, and a larva found in its bed. In July 1898 a mature beetle issued.
C	26½ inches long and 1½ inches in diameter	June 5 to July 13, 1897	Dissected on December 29, 1897, when twelve beds were found, each containing a full-grown larva. These were covered over again. On July 16, 1898 (not having been examined for more than a week), on opening the sack nine live piniphilus were got, their flight holes easy to see. By July 25 other five had issued.

A tabular record of the successive pines used in the piniphilus

experiment will still further prove the continuance of the egg-laying.

No. of Pine.	No. of Beetles.	How long Beetles on Pine.	Proof of Egg-laying.
2		October 12, 1896, to June 21, 1897	New brood issued September 1897.
3	6	June 21 to July 7, 1897	Larvæ on dissection.
4	5	July 7 to July 28, 1897	Dissection on December 4 showed larvæ in beds. Before the end of the first fortnight of July 1898 a number of beetles had issued. On July 22 and 23, 15 more. ,, ,, 25, 4 ,, ,, ,, 29, 6 ,, ,, ,, 31, 1 ,, ,, August 4, 3 ,, ,, ,, 13, 2 ,, ,, ,, 25, 1 ,,
5	4	July 28 to August 28, 1897	In beginning of August 1898 a new brood.
6	3	August 28 to October 2, 1897	

On 2nd October 1897 the three piniphilus were placed on a new pine, on which they remained feeding till the middle of November, when they proceeded to their second hibernation. On 19th March 1898 I found them above ground again feeding on the plant. They were at this time twenty months old.

P. piniphilus then resembles *P. notatus* in its long life as imago and in the continued egg-laying. The generation, following one cycle, is at the most a yearly one, even with the unfavourable condition of development being retarded by the intervention of winter.

Pissodes pini (L.).

Description.—This beetle measures ⅜ inch in length, and is red-brown to black-brown in colour, with sparse yellow scales on both upper and lower surfaces. The punctured thorax has a fine raised middle line; its posterior corners are right-angled, and the

hind edges show scarcely any sinuosity. In front of the elytra are two yellow spots on each side.

Behind the middle of the elytra there is a small continuous transverse band composed of yellow scales compacted together. There are rows of long deep pits down the wing covers.

Life history.—The larger brown weevil, which is found in the centre and the north-east of Scotland (and, according to Fowler, also in Northumberland), lays eggs on old stems of the genus Pinus, Scots pine and the Weymouth pine figuring most largely in notices of attack. The thinner parts of the tree are not neglected by the females; indeed, Altum, generalising from his experience with pini, proclaims that in the first instance it is the upper, thinner parts which are attacked, and later in the progress of the attack the lower thick parts. In my experiments I had egg-laying on perfectly thin twigs. In one case, where I had given a pine log for breeding material, and placed alongside of it a small three-year-old pine, eggs were laid in the latter, and after larval feeding the pupal beds were formed in it; I also got such beds on thin side roots an inch or more below the soil.

Spruce also is sometimes used for egg-laying.

A varying number of eggs are laid in a hole bored by the female in the bark. The larvæ start from their common hatching place and bore out in all directions, the tunnels, however, running chiefly in the long axis of the stem. In one case Altum counted no fewer than thirty of these tunnels starting from one point. The tunnels are long (I have found specimens up to a foot long) and winding, and they often cut one another. The pupa beds, with their characteristic covering of wood chips, are made in the outermost layers of the wood.

It is a practical point worth emphasising that the beds may be quite into the wood.

While examining a pine in the pole stage from Aberdeenshire, I came on the work of pini. Having peeled away the bark, it was easy to trace the progress of the larva by the frass. This I removed with my knife, including the thicker mass at the end, where one might have expected to find either larva or pupa, but neither was seen. Instead was a round hole neatly plugged with sawdust. The grub had bored into the

wood first of all transversely, and then in the longitudinal direction. All behind it was sawdust, and the grub itself was lying in a bed protected by the outer layers of overhead wood. In such a position the larva might easily have attained its full development up to the imago stage, in spite of the tree having been stripped of its bark.

Experiments.

In the winter of 1897–98 I obtained some sections of Scots fir from Aberdeenshire, and on determining that these were infested with the larvæ of *P. pini*, I placed them in a sack in one of the hot-houses at the Royal Botanic Garden.

In the beginning of March 1898 the adult beetles began to come away, and continued to issue until 20th May.

With this material I carried out some experiments in order to compare this, the largest of our British Pissodes, with the other two as regards generation, length of life, and continuance of egg-laying.

Pine 1.

On 2nd June 1898 a healthy and vigorous growing 6-feet Scots pine was uprooted, and replanted in a large tub. The whole was in the usual way surrounded with a muslin sack, and on account of the size of the pine placed (in order to avoid accidents) in a little outhouse at the Royal Botanic Garden. The door of this outhouse stood constantly open, and the weather conditions were the same as if the pine had stood exposed save that it received a certain shelter from the wind and rain. The pine was watered at intervals.

Eighteen *Pissodes pini* were introduced, and remained on the pine until 29th July. The pine was soon studded all over with drops of resin (which ultimately solidified so that the branches were covered with little whitish balls), which had oozed out from the punctures made by the feeding beetles.

In the month of August, when examining the pine here and there on the stem I noticed the bark swollen, and on such places being tested with the finger they 'gave.' Dissection at such places showed that the swelling indicated the path of the feeding larva.

On 12th October 1898, for convenience I removed this pine from its tub and sawed it up into pieces, which were placed in a muslin bag, over which water was thrown at intervals. The spring of 1899 passed without any issue of imago, but on dissecting in June I came across pupæ in their beds, and so might soon expect escape.

On opening the sack on 9th July 1899 (the sack not having been examined for some days) I found a number of *P. pini* had issued, and were crawling about the bag; altogether twenty-nine had issued.

By the end of the second week other 45.
" " third " 31.
" " fourth " 4.
In August other 14 issued.

Pine 2.

On 29th July 1898 I placed thirteen pini on a thick piece of a freshly-felled full-grown Scots pine. This was to serve as breeding material, and in order to supply material on which the pini might feed and so continue to live, I enclosed in the same muslin sack a small three-year-old Scots pine. The pini were removed in this experiment on 30th August 1898. The thick log of freshly-cut pine was very freely used for egg-laying, and dissection after a time revealed feeding larvæ. The first imago of the new brood issued on 24th July 1899. The beetles came away very rapidly. Before the end of July fifty had issued, and by the end of the first week of August other twenty-three.

Pine 3.

On 17th April 1899 I took five *P. pini* of the brood that had issued with one in the spring of 1897, and which had hibernated in 1897–1898, and placed them on a muslin enclosed pine. In the course of the summer breeding was attested by the presence of feeding larvæ. The spring of 1899 was very cold, and this, I think, impeded development. The five pini were allowed to remain on the pine until 27th May, when they were removed. It

was not till the first week of September 1899 that the earliest beetles of the new brood issued, the escape continuing until 26th September.

In all the three experiments the generation is seen to be an annual one.

Length of life in imago stage.—In the case of *P. pini*, also, a long life has to be chronicled. The imagos which issued (after the pupal stage) in March and April of 1898 lived and bred during this year. In November they proceeded to hibernation, reappearing above ground again on 11th March 1899. The five mentioned in Experiment 3 continued to live and lay eggs during 1899, hibernation following in November. On 9th March 1900 I undid the muslin sack that surrounded the pine in connection with which the five pini were hibernating, but could find no beetles. The pot had split in two, and I was afraid of losing the beetles if they should reappear. I therefore decided to look for them in their winter retreat, and on removing the surface soil carefully I came on a male pini which, on being taken up into the warm hand, soon started to move actively about. This beetle was now two years old, and had hibernated twice.

As to when the imago may be got, there was no month in the whole year save January and December when I did not find feeding imagos on my plants. It was very interesting to me to find *P. pini* feeding in one case even in the month of February. This was on a pine where I had ten hibernating pini of a brood of 1899. During some mild weather at the end of February 1900, I had the curiosity to open the sack and examine this pine, when I found that the beetles, tempted by the comparatively high temperature, had left their winter quarters and were feeding on the plant.

General Conclusions.

1. The Pissodes have a remarkably long life in the imago stage. This long life is characteristic of both sexes.

2. Copulation and egg-laying are not single acts, which, once accomplished, terminate the life of the individual, but both may be often repeated. The same individuals which have paired and bred

in one season may, after hibernation, still further proceed to a new season's reproduction.

3. Hibernation takes place in the month of November, and in a season of average temperature ends in March; in exceptionally mild weather even earlier.

4. Egg-laying takes place in all months from April (in a very favourable season even in March) to September inclusive.

5. As adult beetles may be met with during all this period, the length of time necessary for individual development loses some of the significance that up till now has been assigned to it in relation to exterminative measures, because a comparatively limited flight-period being disproved, corresponding limited and definite swarm-periods can no longer be relied on.

6. Still, limiting our view to one cycle and the earliest laid eggs of that cycle, the generation is typically a yearly one.

7. As the first imagos issuing in the summer as a result of eggs laid earlier in the same year are not immediately able to proceed to an efficient copulation, but require some time for ripening, there is little likelihood of there being in our climate two generations in direct descent in one calendar year.

On these conclusions, and the knowledge derived from the breeding and observation of the species, we found the following

Preventive and Remedial Measures.

The great means the forester has in proceeding against these pests once they have got to work is the preparation of catch-trees or decoy stems. These will be sickly plants, or trees left here and there in nursery or plantation; or plants can be artificially weakened and left standing, or an older tree can be cut down and allowed to lie as a breeding place. In consequence of the long-continued life and egg-laying, such trap-plants must be arranged and visited and renewed at intervals throughout the whole year from March till October.

These trap trees will be barked or removed before the enclosed brood has reached maturity and their contents in the shape of larvæ or pupæ destroyed.

My experience is that where full-grown larvæ have been exposed to the light and weather by a stripping of the bark, and a removal of the bed coverings, they do not complete their development, yet it is safer not to give them the opportunity. It should not be forgotten, especially in the case of *P. pini*, that the full-fed larva or the pupa may be protected by the wood under the outermost layers of which they may have bored.

Where notatus is plentiful, collecting the imagos would be a useful measure. This plan could certainly be adopted in nurseries with good results. The beetles would require careful looking for, however, owing to their protective coloration, but favourite places for them are below the whorls, at the bases of the bifoliar spurs, and between the buds. I have pointed out that imagos may be found during many months, and new imago issue also, yet the intervention of winter will give rise to a certain seeming periodicity of imago appearance. Collecting, then, will probably be most successful in the springtime, when the overwintered beetles and the earliest escaping ones renew or proceed to their egg-laying; and also from August onwards, when escape will be at its height.

Where the beetles have not yet got a footing, a timely and vigorous rooting out of all suppressed or sickly pines will go far to prevent injurious attack.

As guides denoting attack we may mention—

(*a*) The bead-like drops of resin that issue from the wounded bark.

(*b*) The drooping of the plants, with a reddening of the needles.

(*c*) The little proboscis puncturings.

(*d*) Broken twigs.

(*e*) At later stages before escape, in young or smooth-barked parts, on the fingers being pulled over the bark little risings may be felt or little ridges may be seen. On cutting into these it will be found that they mark the place of larval tunnel or pupa bed.

Natural aids in checking increase of the pests will be forthcoming from parasitic insects, and from birds. From notatus and piniphilus-attacked material, I have bred out many parasitic Ichneumonidæ, and I have found silver firs 'holed' all down the stem by woodpeckers which had wounded the trees for the enclosed larvæ and pupæ of *Pissodes piceæ*.

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(8) NÜSSLIN, *Forstlich-naturwissenchaftlichen Zeitschrift*, 1898.

(9) ALTUM, *Forst-Zoologie*, vol. iii. p. 210.

(10) NITSCHE, *Lehrbuch der Mitteleuropaischen Forstinsektenkunde*, p. 381.

The Biology and Forest Importance of *Scolytus* (*Eccoptogaster*) *multistriatus* (Marsh). By R. Stewart MacDougall, M.A., D.Sc. *Communicated by* Professor Cossar Ewart.

(Read June 4, 1900.)

The Scolytidæ is a family of small roundish tetramerous beetles characterised by the fact that the female beetle enters bodily the tree or plant for her egg-laying, the eggs being generally laid in little notches cut out in the sides of the mother gallery. With some species, however, the eggs are laid all together in a bunch. The grubs are whitish, wrinkled and legless, and have brown scaly heads. The close resemblance to each other of the grubs of the various species renders the determination of the species from larval characters extremely difficult, if not impossible, but the figures or patterns presented by the mother gallery and the larval galleries in relation to it are in general so highly characteristic, that with these and the name of the host plant one can generally determine the species.

The family Scolytidæ numbers in it some of the very worst insect enemies of our woods and felled trees. Some do harm as imago by gnawing the roots of conifers; some, both as imago and grub, attack the bast of grown conifers; others, again, like *Hylesinus piniperda*—that scourge of our pine-woods—do harm as newly-issued imagines by tunnelling into the young shoots, and later, both as imago and larva, boring their galleries in the cambial region, interfering with the conduction of sap, and weakening or killing the tree; while members of still another group bore into the wood and render it useless for technical purposes.

Among the six species of Scolytus given by Fowler as British, we have enemies of the birch, oak, and elm. Two species attack elm, viz., *Scolytus destructor*, Oliv. (*Geoffroyi*, Goetze), the larger elm bark beetle, and *Scolytus multistriatus*, the smaller elm bark beetle.

As regards insect and work, the two may be distinguished thus:—*S. destructor* is larger, measuring 4 to 6 mm., *multistriatus* being in length only 3 to 3½ mm. The larvæ of *destructor* are also larger, hence the mother gallery and the resulting larval tunnels are also of greater circumference. The larval galleries of *S. destructor* from each mother gallery are not so numerous nor so close together as those of *S. multistriatus.*

Scolytus multistriatus.

The beetle is black or dark brown, and glossy, the antennæ and legs paler. The thorax is longer than broad and very finely punctured, the punctures on the flat part being finer and not so thick as those at the sides. The brown elytra, somewhat narrowed behind, show many punctured striæ. From the posterior margin of the second abdominal segment there projects a moderately long, strong spine, backwardly directed. In the male the forehead is somewhat compressed, and bordered at the sides and behind with greyish-yellow little bristles. In the female the forehead is somewhat arched and lacks the bristles. Length, 3 to 3½ mm.

After fertilisation and the boring into the bark of the elm, the female gnaws out in the cambial region a gallery, longitudinal in direction. This gallery cut out in the youngest wood-layer varies in length between one and two inches, measurement of some of the galleries in my experiment giving 1⅛ in., 1¼ in., 2 in. In shape the gallery resembles a miniature golf club, the head of the club marking the place of entrance and start. Along the sides of this neat gallery, the mother cuts little notches at equal distances from each other, and in each notch an egg is laid. The legless, whitish, brown-headed grubs on hatching out proceed to gnaw their tunnels at right angles to the parent gallery. These tunnels, crowded together, are cut chiefly into the bark, but where the bark is thin their course can be traced also on the outermost layers of the wood. As the tunnels run out from the parent gallery, they cease to be at right angles, but bend, some upwards some downwards, while the width of the tunnel keeps increasing with the growth of the grub. At the end of the larval tunnel (some of the tunnels in my specimens were 2½ inches long)

the full-fed grub pupates in an oval bed hollowed out in the bark, whence later, after pupation, the imago bores out through bed and bark, the flight holes on stem or branch from which a brood has issued resembling a number of small shot-holes. If one examine the beetles in their beds soon after they have ceased to be pupae, their colour is light-brown yellow, with dark glossy heads.

While continental writers were unanimous on the point of *multistriatus* being a late swarmer, not appearing, it was said, until a summer temperature had been reached, there were no experimental records as to the length of time necessary for the completion of the life cycle, and partly to make certain of this and partly to determine whether *multistriatus* would attack (and be successful in attack) a healthy tree, I undertook my experiment.

Previous to the experiment, I had recorded several observations of this beetle in my notes. Thus in Munich, in the autumn of 1894, several elm logs on dissection showed larvæ of *multistriatus.* These logs after being kept in water for some time were placed in a room, where they remained quite dry until the spring of 1895, when, again, they were placed in water. At the end of June and in the first days of July, the beetles began to issue from the logs.

Again, in Munich, in the laboratory of Professor Pauly, I noted escape of beetles as follows :—

Date of Issue.	Number of Beetles.	Date of Issue.	Number of Beetles.
1895, July 1	144	1895, July 11	7
„ 2	16	„ 12	4
„ 3	20	„ 13	1
„ 4	13	„ 14	2
„ 5	4	„ 15	1
„ 6	8	„ 16	1
„ 7	5		

With some of this material I started an experiment in Munich, and in July 1896 brought with me to the Royal Botanic Garden, Edinburgh, from Munich, the sack containing the prepared pieces of elm and the beetles. From pressure of work, however, I was unable to attend further to the matter. In the autumn of 1897, when removing the pieces of elm from the sack in which they had been standing since July 1896, I noticed them covered with

flight holes, indicating that some time in 1897 there had been an escape of a new brood of beetles, and that my experiment would have been successful had I had leisure to attend to it.

In February 1898 I took out one of the branches from which a brood had issued, and was dissecting it with a view to making a museum preparation of the work of *S. multistriatus*, when I came upon some living larvæ. These must, I think, have come from eggs laid by some of the 1897 beetles, which thus appear to have used for breeding purposes the very same branch in which they themselves had been bred. This branch—22 inches long by $1\frac{3}{4}$ inches in diameter—cut in July 1896, had been paraffined at the cut ends to prevent excessive loss of mixture, but by 1897 must have lost its freshness and been dry and dead.

In order to allow the larvæ present in the half-dissected specimen of elm to attain their full development, the branch was placed in a cotton sack, and exposed in the Garden to all weathers. On 15th July 1898 beetles began to issue, and from this dead dry branch I obtained on

1898, July	15 ...	4	*multistriatus.*
„	18 ...	2	„
„	23 ...	3	„
„	24 ...	3	„
„	27 ...	2	„
„	29 ...	1	„

With this fresh supply of *multistriatus* I started a new experiment.

Method of Experiment.

Two branches of *Ulmus campestris*, freshly cut in the Royal Botanic Garden, each measuring 2 feet long by $2\frac{3}{4}$ inches in diameter, were placed in a cotton sack, after being paraffined, *i.e.*, the cut ends of the branches had been dipped in melted paraffin, which when solidified had formed a crust over the cut surfaces. This coating of paraffin, by causing retention of moisture, kept the branches fresh for a much longer time than they would have remained so without the treatment. To the sack containing the elm branches eleven *S. multistriatus* were added between July 15 and July 19, 1898. The eleven were placed in without their sex being determined, as determination of these small beetles, with a

lens, meant a handling of them such as might have risked their life, and as my material was not plentiful, I was unwilling to run the risk of loss.

Examination of the two branches on 5th August revealed several entrance holes in the bark, a slight outflow of sap marking the place of the beetles' entry. Three of the eleven beetles were lying dead in the bottom of the sack. On 20th September other two dead beetles were found, and a live one, which I kept out.

On 10th February 1899 one of the branches was carefully dissected, an entrance hole being followed up, when I found that at this place a mother gallery had been made and eggs had been laid, the larvæ—exposed by the scalpel—having started to gnaw out their galleries. These larvæ were very small, and had not progressed far from the mother tunnel. As the year went on constant examination was made regarding the imago issue, and at last, on 13th July, the first new beetle issued. Here is the record of escape from these branches:—

Date of Issue.	Number of Beetles.	Date of Issue.	Number of Beetles.
1899, July 13	1	1899, Aug. 5	4
„ 14	2	„ 6	3
„ 16	2	„ 7	1
„ 18	1	„ 8	1
„ 20	2	„ 9	1
„ 21	3	„ 10	1
„ 22	1*	„ 11	6
„ 24	3	„ 13	7
„ 26	1	„ 15	2
„ 29	4	„ 19	1
„ 30	3	„ 21/22	12
„ 31	1	„ 24	1
Aug. 3	4	„ 26	1
„ 4	3	Oct. 13	2

Dissection of the branches on 13th October showed several full-fed larvæ in their beds.

From the foregoing experiments and observations it will have been noticed that the earliest time of issue for the adult beetle has always been June or July. The generation of the June or

* A larva and a pupa lay not far from the exit hole of this beetle.

July beetles is an annual one, the larvæ from the eggs of these first beetles passing the winter as larvæ and completing their growth in the spring and early summer of the next year in time to allow of preparation and escape of imagos in July.

With the material thus obtained in July and August, an experiment was arranged to test whether or no *multistriatus* was able to attack successfully and breed in a perfectly healthy tree.

Method of Experiment.

A large cotton sack in the form of a sleeve open at both ends was slipped over a vigorous branch of a healthy *Ulmus campestris*, the branch, of course, not being severed from the tree. One end of the sleeve was securely bound round the branch, and the other end, after the introduction of the beetles, likewise secured. The sleeve was wide, and by means of thin stakes it was kept from touching the branch, except at the secured ends.

The above was done on two different branches, the sleeve on Branch I. holding twenty-two *multistriatus* introduced between July 15 and July 30, 1899, and that on Branch II. holding twenty-three *multistriatus* introduced between August 13 and August 26.

At different times the 'sleeves' were opened, and up to November live *multistriatus* were seen crawling over the branches. Examined in July of 1901, and later, the two branches were both alive and showed quite green on dissection. In neither case did the beetles succeed in rearing a brood.

It would seem, then, that *S. multistriatus* alone and unaided is not a formidable enemy of our elm trees, although in conjunction with *S. destructor*, the larger elm beetle, and seconding the work of the latter, it might have considerable importance.

In cases where *multistriatus* was proving troublesome, the attacked trees should be felled, and the branches containing the enclosed brood burned.

Perhaps a useful measure would be the preparing of sickly trees or branches as traps for the beetles to lay in, these to be peeled before a sufficient time had elapsed for the larvæ to have completed their development, and the bark burned.

Note on the New Star in Perseus. By The Astronomer-Royal for Scotland. (With a Plate.)

(Read March 4, 1901.)

We are again indebted to Dr T. D. Anderson of this city for the announcement of the discovery of a new star, which was first seen by him at $2^h\ 40^m$ a.m., G.M.T., on Friday, the 22nd February.

Shortly after eleven o'clock on the forenoon of that day Dr Anderson came to the Royal Observatory and communicated the exact particulars of the startling phenomenon. The approximate position of the star in the heavens was R.A. $= 3^h\ 24^m\ 25^s$, Decl. $= +43°\ 34'$; it was of the 2·7 magnitude, and of bluish-white colour. Telegrams were at once dispatched to the Royal Observatory, Greenwich, and to the International Central Bureau for Astronomical Telegrams at Kiel for general distribution to the observatories of the world. To make assurance doubly sure, special telegrams were also sent to a few distinguished spectroscopists.

The magnificent spectroscope, presented to the observatory by Lord Crawford, and specially constructed by Messrs T. Cooke & Sons of York for stellar spectroscopy, was at once mounted to the 15-inch refractor, and everything prepared, as far as possible, for observation. Fortunately the sky partly cleared in the evening, when, at $6^h\ 30^m$ p.m., I had the great pleasure of inspecting the star with the 6·3-inch Simms' refractor and a small direct vision prism. The first impression was in a certain sense disappointing, as the spectrum showed none of the striking peculiarities so conspicuously displayed in the case of Nova Aurigæ, which, it will be recollected, was also discovered by Dr Anderson. The spectrum was brilliant indeed, but apparently absolutely continuous from the red to the extreme violet; a fact which was confirmed by Mr G. Clark. The first view with the larger instrument gave no further information, and it was only on very careful inspection that Dr Halm noticed about half a dozen delicate absorption lines, and, in addition to these, two hazy dark bands, closely accompanied in each case by indications of brighter intervals on the less refrangible side. The wave lengths of these bands were

501 and 486, and thus would seem to agree with those of the two principal nebula lines.* The positions of these features were measured as satisfactorily as their faintness permitted.

Several of the dark lines I was able to confirm, but thickening haze prevented further observation. On this night Mr Heath secured a photograph of the star with the 24-inch reflecting telescope through shifting clouds. The night of the 23rd was unfortunately overcast, except for a very short interval at about eight o'clock, when Dr Halm and Mr Clark saw the Nova and estimated its brightness, which was found to exceed that of Capella by a fifth of a magnitude. An attempt at viewing the spectrum with the 15-inch refractor led to no result.

The sky was completely overcast on the 24th, 25th, and 26th. On the 27th the weather was more favourable, although observations could only be made through rifts in the clouds. The whole character of the spectrum had, in the interval, undergone a profound change, and now resembled indeed that of Nova Aurigæ when at its greatest brilliancy. Besides the bright bands suspected on the 22nd, which had now increased so much in brightness as to become the most prominent feature in the spectrum, the C-line of hydrogen had blazed out with great brilliancy. This line had been specially looked for on the 22nd, but no trace of it could then be distinguished. Nearly all the bright bands were of considerable width, being, in fact, in general not less than three times the width of the slit used. It required no considerable optical means to bring out the special characteristics of the Nova type, for the whole spectrum was beautifully shown on applying a tiny direct vision prism and cylindrical lens to the eye-piece of the finder, which has an aperture of only $3\frac{3}{4}$ inches.

Owing to the repairs of the dome, now in progress, the automatic driving of the telescope was to some extent deranged. In spite of these untoward circumstances, Dr Halm and I succeeded in securing a fair number of measures of the principal spectroscopic features.

* Note added March 15.—While the band at 486 $\mu\mu$ is undoubtedly due to hydrogen, which is also present in the spectrum of the nebulæ, that at 501 $\mu\mu$ does not appear, as later measurements showed, to coincide with the chief nebula line at 500·7, but is probably identical with the chromospheric line 501·8 due to iron.

Somewhat similar observations were made on March 1st, but the impression was gained that the dispersion employed was too great for the star's diminished light. Accordingly a 30° prism of very transparent flint glass by Salleron was adapted to the spectroscope.

On resuming work on the 3rd, under favourable atmospheric conditions, this change in the apparatus told with full effect. All the larger bright lines were well defined, each one with an attendant deep black line on the more refrangible side.

This very favourable night afforded a large number of satisfactory measures, which still await final reduction. The chief results may, however, to some extent be summarised as follows:

The spectrum seems to be due to two media, one of which emits light of a limited number of definite wave-lengths, and must therefore be considered as gaseous. The continuous background may reasonably be attributed to matter of a liquid or solid constitution. The dark lines are then the effect of absorption on the part of the same kind of gaseous matter that yields the bright spectrum, only with this difference, that the absorbent medium must be of lower temperature than the body producing the continuous spectrum, and that it is being carried towards us at a very high velocity. It is not at all necessary that this absorbent layer should be of great thickness, provided it is of sufficient density. The relative difference of velocity of the two bodies is quite stupendous, the reduction of the observations so far yielding the enormous value of 800 miles per second. It is certainly remarkable that this Nova should show a displacement of nearly the same amount and towards the same side as Nova Aurigæ. It is not altogether inconceivable, however, that the two stars may have something in common as regards their origin, as they are both in the Milky Way, and not more than 30° apart.

The accompanying drawing made by Dr Halm, which was shown at the meeting, represents the spectrum as seen in our instrument on March 3rd. The intensity curve is based on estimates of brightness of the bands made at the same time.

A number of photographs of the violet part of the spectrum have been secured by Mr Heath, using an object glass prism in front of the 6·3 inch equatorial. Unfortunately it has not yet

been possible to determine the wave-lengths of the lines shown on these plates, but the general character of the spectrum seems to agree with that of the visual part.

In the *Times* of March 1st Miss Agnes M. Clerke propounded the hypothesis that the broadening of the lines in the spectrum of the new star might be due to the influence of a powerful magnetic field, and that in this case their light would be polarised, so to speak, in "sections," thus affording an instance of the well-known "Zeeman phenomenon." At the same time Miss Clerke indicated how the question might be at once decided with the help of a Nicol prism. Last night afforded a singularly favourable chance for making this interesting experiment. Accordingly, at a time when the sky was perfectly clear, and the spectrum was consequently seen to the best advantage, the chief lines were carefully examined with a square-ended polarising prism by Dr Halm and myself. No trace of polarisation was, however, visible; on the contrary, the bright lines could be clearly seen of their full width in all positions of the prism. To whatever cause, therefore, the extreme width of these lines may be due, it is not to the one so ingeniously suggested by Miss Clerke.

Regarding the brightness of the star, the following notes may be of general interest:—

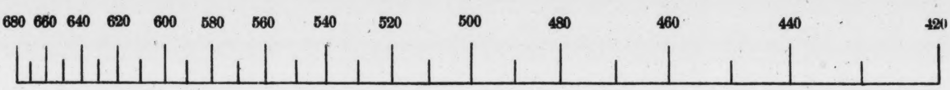
Feb. 19.—Prof. Pickering photographed that part of the heavens without obtaining a trace of the star, which he considers must therefore have been fainter than 11th magn.

				m.		
Feb. 21,	14^h	40^m	M.T.Gr.,	2·7	Anderson.	
22,	6	58	,,	0·7	Copeland.	
,,	8	10	,,	0·5	,,	
23,	8	10	,,	0·0	Halm and Clark.	
27,	11	15	,,	1·6	Copeland;	decided yellow.
Mar. 1,	11	0	,,	2·3	,,	
2,	11	40	,,	2·2	,,	
3,	12	25	,,	2·0	,,	orange red.

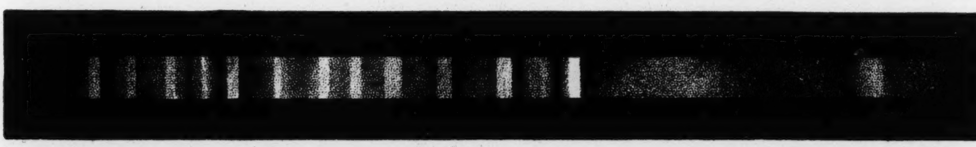
From the 19th to the 23rd the star must have increased in brightness at least 25,000 times (25,120).

On the other hand, in the interval between the 23rd of February

SCALE OF WAVE LENGTHS.



SPECTRUM OF NOVA PERSEI, AS SEEN ON MARCH 3RD, 1901.



INTENSITY CURVE.

A. RITCHIE & SON, EDIN^R.

and the 3rd of March (or in eight days), it must have lost fully $\frac{5}{8}$ths of its light.

As bearing on the sudden appearance of the star, we have an interesting note from Mr W. B. Dodd, of Whitehaven, who independently discovered the Nova on the night of February 23rd. On the night of the 21st, some three hours before the star was first seen by Dr Anderson, Mr Dodd's attention chanced to be directed to the constellation of Perseus. He writes:

"Occupied with Perseus at 11.45; tried to get the telescope pointed on ϵ Persei, but the star had got too low for the stand I was using. I glanced across the constellation to Algol, and remembered that there was no star as bright as either of them [ϵ or β Persei] in the space between."

Additional Note on the Ultra-Neptunian Planet, whose existence is indicated by its action on Comets. By Professor George Forbes, M.A., F.R.S. (With a Plate.)

(Read May 6, 1901.)

The history of research in this planet is briefly as follows:—

In 1879 Professor Newton enunciated the proposition that if the elliptic orbits of comets have been changed from parabolas by planetary perturbations, then the probabilities are in favour of the comet's position at the time becoming the aphelion position of the new orbit. This explains why the aphelion distances of so many comets agree with the mean distances of Jupiter and Neptune respectively.

At the meeting of the British Association when this was announced, I stated that if this be true there are certainly two undiscovered planets beyond Neptune, one of which is at a distance from the sun about 100 times the mean distance of the earth from the sun.

In 1880, on 16th February, I made a communication to the Royal Society of Edinburgh,* referring to seven comets whose aphelia were calculated to be at this distance, and describing an attempt to determine the present position of the new planet on the supposition that it occupied the longitudes of the several aphelia at dates when the comets were at those aphelion positions. Mr Isaac Roberts made a search by photography but did not find the planet, possibly owing to my having indicated for his search an area that was too limited.

These calculations have lately been revised by me, use being made of every elliptic orbit in Galle's recent Catalogue (Cometen-

* A short abstract appeared in the *Proceedings*. I printed privately 100 copies of the full paper, which were distributed to observatories and astronomers who applied for them. The present Astronomer-Royal, who at that time edited *The Observatory*, published the full paper in the issue of that journal for June 1880. The perturbations of Uranus by the new planet were discussed in a paper read to the R.S.E., 1880, May 17th. Further particulars were given to the R.S.E., 1881, January 17th. Both of these appear in the *Proceedings*.

bahnen, 1893) which could throw light on the subject. The results are interesting, and generally confirm the conclusions arrived at in 1880 as to the probable position of the new planet. The whole of the work was gone over, reasons were found for altering some of the data, an error in one of the calculations was discovered, and a comparatively recent comet was added to the list. Yet the final position assigned to the planet was unchanged.

The present 'Note' comes from the discovery of a remarkable confirmation of these results. It is well known that the comet of 1556, which has generally been looked on as a return of the comet of 1264, did not reappear in 1848 as was expected. In fact, it seems to have disappeared as completely as did Lexell's comet of 1770 by the attraction of Jupiter upon it when in aphelion.

The longitude of the aphelion of comet 1556 was 99°·24′ in the year 1696, and its distance from the sun was 88 times that of the earth. Now, I find that if my published results be correct the longitude of the new planet in 1696 was 112°, its distance from the sun being 100 times that of the earth. From this it appears to be highly probable that the non-return of the comet was due to its deflection at aphelion by the new planet.

Anyone who has read Laplace's analysis of the action of Jupiter upon Lexell's comet * must realise that if Jupiter's longitude had been unknown it might have been found by the action upon the comet. So also in this case we may deduce conclusions which must be true if the comets 1264 and 1553 were identical. And the first conclusion is that the longitude I have assigned to the planet which we know to be at 100 times the earth's distance from the sun is not far wrong.

The latitude of the comet 1556, when in aphelion, was 30°. Hence its distance from the planet was very much greater than is the case with ordinary cometary perturbations considered by astronomers. On the other hand, such perturbations are important only for a few days or weeks, while in the present case the influence remains of the same order of magnitude for nearly two hundred years.

It becomes then a matter of great interest to examine, generally

* *Mecanique Celeste*, vol. iv., pp. xviii. and 223, etc.

in the first place, the nature of these perturbations on various assumptions as to the mass of the planet. If it be found that the perturbations would not suffice to prevent a return of the comet, in a moderately changed orbit, except on the assumption of a mass so great that its influence on other planets could not have escaped notice, then we may be sure that, if comets 1264 and 1556 were identical, the comet must have returned as an unrecognised comet in an altered orbit. If we can identify comet 1556 with such a comet seen in the last half century, a beautiful problem presents itself: Given an orbit transformed into another given orbit by a planet of unknown mass in a position approximately known, determine the mass and exact longitude of the disturbing planet.

The estimation of the general character of the perturbations is facilitated in the present case by the following considerations:—

1. The aphelion radius vector (or the line of apsides) is very nearly in the line at right angles to the line of Nodes, being only $4\frac{1}{2}°$ from it.
2. The comet's aphelion being 88 times, and the planet 100 times, the mean distance of the earth from the sun, and the angle between the radii vectores of the two bodies at the aphelion being 31°, it follows that, at and about the time of greatest disturbance, the perturbations by the planet are almost entirely perpendicular to the plane of the comet's orbit, so increasing the inclination and retrograding the line of Nodes.
3. At any other position of the planet where there is any component in the plane of the comet's orbit, the action is such as to increase the longitude of Perihelion

I have made a preliminary computation of the general character and amount of these perturbations, and find that if the new planet have the same mass as Jupiter, the orbit of this comet would not be so seriously affected as was that of Lexell's comet by Jupiter; but the plane of the new orbit would be inclined to that of the old one at about 5°, so that the longitude of the Node would be retrograded about 12°, and the inclination of the orbit to the Ecliptic would be increased by about 3°, and the longitude of Perihelion would be advanced slightly.

But the number of comets affected by this new planet is so

large that in all probability the new planet has a greater mass than Jupiter. If the new planet be several times the mass of Jupiter, the orbit of the comet of 1556 might be so much disturbed as to render the comet on its return unrecognisable, if the existence of the new planet be ignored.

A careful examination of all the comets in Galle's Catalogue, to which elliptic orbits have been assigned has convinced me that no one of them is the lost comet 1556.

At the same time, if the new planet had deflected the comet so far as to prevent its return up to now, the planet must have a mass so great that its influence on planetary orbits would ere now probably have been detected. It is therefore desirable to search among the comets to which elliptic orbits have not yet been assigned, to see whether any one of them may be the lost comet 1556.

Upon making this search, I found that Comet 1844 iii., which has been assumed to have a parabolic orbit, would, if its orbit were elliptic, have its aphelion in longitude 116°, while Comet 1843 ii. would have its aphelion in longitude 115°, and no other comet in the whole of Galle's Catalogue can possibly be identified with Comet 1556. It is to the first of these, 1844 iii., that I wish in the first place to draw attention. According to the ephemeris published by me, the aphelion longitude of this comet was occupied by the planet in the year 1705, *i.e.*, about the same time as the comet itself. Both Encke and Cooper (*Cometic Orbits*, p. 173), besides others, have noticed a similarity between this comet and 1556. I find that if this comet be moving in the disturbed orbit of 1556 the Node has been retrograded considerably, the inclination has been increased, and the longitude of Perihelion has been advanced. In all these points it agrees with the character of the perturbations that we should expect the new planet to produce, as stated above. Also the line of intersection of the two orbits is near their aphelia, and is approximately in the position suggested by a preliminary examination. Only the latitude of Aphelion is smaller than would be expected on any moderate assumption as to the mass of the planet. This is the only apparent discrepancy that appears in the preliminary investigation. In all other particulars the orbit of Comet 1844 iii.

appears to be the orbit of the Comet 1556, perturbed by a planet considerably larger than Jupiter,* situated at or about the position indicated as to radius vector and longitude in my original communication to the Royal Society of Edinburgh in 1880, according to which the planet is at 100 times the mean distance of the earth from the sun, and is in longitude 181° in this year 1901.

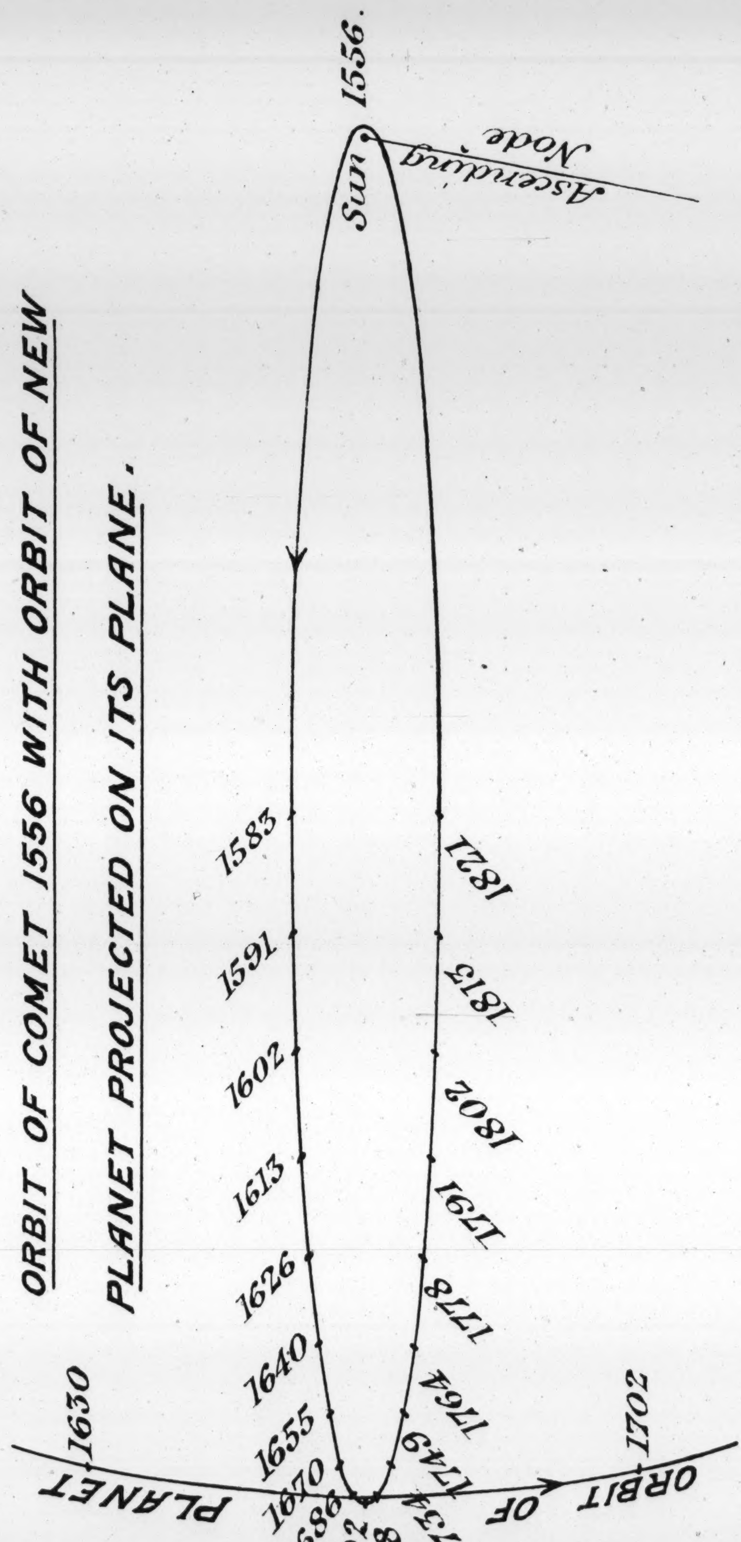
With regard to Comet 1843 ii., if this be a reappearance of Comet 1556, the Nodes have been retrograded, the inclination increased, and the longitude of perihelion advanced, as in the other case. But the latitude of aphelion has not been reduced like the other, but rather increased. Also the Perihelion distance has been increased quite sufficiently to account for the inferior display and the insignificance of its last appearance.

It would be rash to make any further expression of opinion until the calculations have been completed. In the meantime the conclusions certainly arrived at are the following:—

1. The position of the new planet as stated in 1880 is confirmed by a fuller investigation on the same lines.
2. If the comets of 1264 and 1556 were identical, the new planet would produce perturbations whose amount is sensible, and these account for the non-reappearance of the comet in its old orbit, and may lead to further knowledge about the mass and position of the new planet.
3. It is possible that one of the comets, 1844 iii. or 1843 ii., may be the lost comet of 1556, perturbed in its orbit by the new planet; and the re-examination of the 1556 observations, and the computations which I am now engaged on, must throw some light on this question.

* In the paper which I read to the R.S.E. in January, 1881, the perturbations of Uranus by the new planet led me to estimate its mass at a little more than half that of Jupiter.

ORBIT OF COMET 1556 WITH ORBIT OF NEW PLANET PROJECTED ON ITS PLANE.



A. RITCHIE & SON, ED

On Hair in the Equidæ. By F. H. A. Marshall, B.A., F.R.S.E. (With Six Plates.)

(Read June 17, 1901.)

The taxonomic value of hair has long been recognised. The different types of human hair have been made use of as a basis for classification of the varieties of Man by Primer Bey* and many others, while Waldeyer† in his Atlas has described briefly the hair characters of well known members of the Mammalian orders. In the present paper it is proposed to deal with hair within the limits of a single family, that of the Equidæ, and to describe certain peculiarities in the hairs of members of that group, which the author is of opinion are probably of specific value. But before dealing with the hair characters by which the species may be distinguished from one another, something must be said about those of the group as a whole.

The characters by which hairs of different animals can be distinguished from one another, apart from their length, shape, and colour, the latter being of little or no taxonomic value, are the nature of the cuticle, the extent of development of the medulla in different parts of the hair, the relative thickness of the medulla, and the arrangement of the pigment in the cortex. The cuticle presents comparatively slight modifications, and consequently the characters of this layer are not of much value for taxonomic purposes. In the hairs of the different members of the Equidæ it is, so far as I have observed, almost identical, being smooth or only slightly imbricate. In transverse sections it appears little more than a line bounding the cortex on the exterior.

The medulla, on the other hand, shows very great variability in different animals, and the accounts given of it by various writers

* Primer Bey, "Human Hair as a Race Character," *Jour. of Anthropological Institute*, vol. vi.

† Waldeyer, *Atlas des Menschlichen und Tierischen Haare*, etc., Lahr, 1884.

differ widely from one another. Primer Bey, writing of human hair, describes three kinds of hair differing in this character, according as to whether there is a central canal devoid of medullary substance, a canal filled with medulla, or whether the hair is composed of cortical substance throughout. Reissner * refers to the partial absence of medullary substance in some animals, and its total absence in seals and some Chiroptera. Ridewood, † in a recent paper, draws attention to its absence in sloths, and quotes Welcher, who first noticed this fact. Poulton ‡ states that the medulla is wanting in the slender unpigmented base and also in the 'neck' region in the hairs of *Ornithorhynchus*. Henle, § who describes the medulla as a substance consisting as a rule of two rows of cells whose nuclei are flattened transversely, says that this substance is quite absent in the finer hairs, and is not constant in the stronger ones, failing here and there. Other authorities might be quoted to show the variability of the medulla in different animals' hairs.

In all the equine hairs that I have examined, even in the very finest, the medulla is present, though its degree of development is somewhat variable. It is usually absent for a considerable distance, both from the point and from the base of the hair, and may have broken down in an unaccountable fashion in one or more places on the hair shaft. Moreover, it consists, at least in its thickest part, of certainly more than two rows of cells, the nuclei of which can be seen in suitably stained sections. They are not shown in the figures illustrating this paper, which are drawn from unstained preparations. The absence of the medulla at the base of the hair is accompanied in many cases by the absence of pigment in the cortex. This is well shown in the hairs of the Somali zebra, which will be described lower down. Such an absence is invariable in fully grown hairs. Hairs which have not yet grown to their full length retain the medulla to a point much nearer the root. This shows that with the growth of the hair, the medulla

* Reissner, *Beiträge zur Kentniss der Haare*, Breslau, 1854.

† Ridewood, "On the Structure of the Hairs of *Mylodon listai*," *Q.J.M.S.*, vol. xliv.

‡ Poulton, "The Structure of the Bill and Hairs of *Ornithorhynchus paradoxus*," *Q.J.M.S.*, vol. xxxvi.

§ Henle, *Hand. der Eingeweidelehre*, Braunschweig, 1873.

tends to disappear towards the root. The manner of its disappearance is an open question. Mertsching,* after referring to certain statements by Kölliker that the frequent absence of the medulla in coloured human head hairs, and its almost regular occurrence in white head hairs, says that this points to the inference that the formation of the medulla is connected with the turning grey of the hair. This, however, cannot apply to equine hairs. But the colour of a hair to the naked eye is affected by the breaking down of the medulla, such hairs appearing considerably duller and darker. Thus light brown hairs become dull brown.

Speaking generally, then, equine hairs may be said to be characterised by the invariable presence of the medulla to a greater or less degree of development, and by the tendency of the medulla to disappear at irregular intervals, leaving air spaces of all sizes. This latter characteristic appears in Waldeyer's figure of horse hair, but not in his figures of the hairs of other Mammalia.

Another character by which the equine hairs may be distinguished from other hairs, and from hairs of other species in the genus, is the distribution and arrangement of pigment in the cortex. Nathusius † has called attention to the fact, which I have often observed, that in some species of the genus *Equus*, the pigment granules on one side of the medulla may present a different coloration to those on the other side; in other words, that the hair may be striped longitudinally. This character, so far as I have observed, does not hold good for horse hairs, but it is very general in other members of the family. The hairs in such cases are coloured by at least two different sorts of pigment, which have blended unequally on the two sides of the hair. In this connection, it is interesting to repeat for equine hairs some of Sorby's ‡ experiments on human hairs. When brown hairs of the type in which longitudinal striping is common are dissolved in a strong

* Mertsching, "Beiträge zur Histologie des Haares und Haarebalges," *Arch. f. Mikr. Anat.*, Bd. xxxi. 1888.

† Nathusius, "Über die taxionomische Bedeutung der Form und Farbung der Haare bei den Equiden," *Verhand. d. Deut. Zool. Gesellschaft auf der zweiten Jahresversammlung zu Berlin*, June 1892, Leipzig, 1892.

‡ Sorby, "On the Colouring Matters in Human Hair," *Journal of Anthrop. Inst.*, vol. viii.

solution of sulphuric acid, it is frequently found that one or more of the pigments goes into solution which is coloured, as do also the other constituents of the hairs, while another pigment sinks to the bottom undissolved. This result is similar to Sorby's for black human hair, which contains a quantity of brown or red pigment, which colours on acid solution, the dominating black pigment, which causes the hairs to appear perfectly black, sinking to the bottom as a precipitate. This, however, I have not found to be the case with black horse hair, for when this is dissolved in strong acid, after the black pigment has sunk to the bottom, the acid solution remains perfectly clear and uncoloured. When white, or nearly white, horse or ass hairs are dissolved, the solution is also clear, this result agreeing with Sorby's for white human hair. The study, however, of the different sorts of pigment, whether in equine or other hair, and the application of the spectroscope to the problems presented, is the work of the chemist.

In discussing hair coloration, it is well to remember that the tone of colour presented by the hairs collectively on the skin is often quite different to that of the individual hairs when viewed separately through the microscope. This must be due to the blending of the different shades of colour in the general effect. The colour of a hair is commonly supposed to depend on the presence or absence of the pigment granules of different shades in the cortex. This is, of course, largely the case. But there is often in addition a diffuse coloration throughout the cortex, and as above remarked, the colour of a hair is affected not inconsiderably by the degree of development of the medulla, and what is related to it, the presence or absence of air vacuoles in the medullary canal.

A few remarks must be made concerning the shape of equine hairs. Nathusius, in his investigations, made use of hairs from the shoulder region or from the side of the body, and remarks that in these, with the increasing thickness of the hair the cross section becomes more oval and less circular in shape. Thus the most circular sections are those through the medullaless regions near the point and root. In another place,* Nathusius appears to apply

* Nathusius, "Über Haar-Formen und Farben von Equiden," *Landwirtschaftliche Jahrbucher*, Bd. xxvi., 1897, Berlin.

this description to hairs in general. My own observations have shown it to be very generally applicable to the shoulder and side hairs in the Equidæ, but this cannot be said for the hairs of the mane, sections through which are commonly circular throughout the entire length of the hair, the exceptions being, so far as I have seen, certain very long and fine mane hairs of horses and the mane hairs of the mountain zebra. These are elliptical. The flattening, as is well known, is related closely to the tendency the hair has to curl. Thus, in animals with stiff upright manes we should expect to find a circular hair section.

Like Nathusius, whose investigations were almost entirely upon horse and ass hairs, I have employed hairs from the shoulder region. By simply mounting such hairs in balsam, many of the characters can be quite well made out. But I have also employed mane hairs, which from their greater value for taxonomic purposes and the much greater ease in cutting them into transverse sections, are more useful for purposes of comparison. Cutting sections through hairs is always a matter of some difficulty, and not the least part of it is to contrive that the sections shall be transverse. Dr Hepburn has been kind enough to show me the apparatus he has invented and employed for keeping the hairs stretched out during embedding. It consists of two small metal boxes open at their ends and made to fit into each other. The hairs are stretched across the open end of one of the boxes, which on being fitted into the other one, retains the hairs in position. The whole apparatus can then be embedded in paraffin, and the paraffin block containing the stretched hairs can be cut out of the metal box (since the ends are open) after solidifying. I have employed this apparatus for the shorter hairs, but for longer hairs it is just as easy to embed in an ordinary paper box, keeping the hairs stretched across by fastening their ends in holes in the paper. The hairs were cleared in xylol or turpentine before being embedded. I have found paraffin of a melting point of 58° C. the best for embedding in. The sections were cut with a Cambridge rocking microtome at a thickness of 10μ, cleared in xylol and mounted in Canada balsam.

The material employed has been largely provided by Professor Ewart, either from animals in his stud at Penycuik or from skins in his possession; but I have been able to confirm some of my

observations on hairs obtained from animals in the Gardens of the Zoological Society of London. In work of this sort it is desirable, before setting down certain characters as those of the species, to confirm one's observations in as many individuals as possible. Where only one individual of a species is studied, it is easy to fall into the error of regarding certain characteristics as belonging to the species which are really only individual peculiarities. It is also well to remember that it by no means necessarily follows that because hairs possess certain general characters which it is usual to find in the members of a particular family, such as that of the Equidæ, that they must belong to an animal which is a member of that family. On the other hand, it is natural to suppose that the causes which operate in determining a particular form of hair in the members of one family should operate and bring about similar results in the members of a quite different family. It is acknowledged that those who maintained that an extinct animal could be restored by an examination of a single bone went a great deal too far. And so, in the absence of other evidence, to attempt to assign an animal to its genus on the strength of the characters of some of its hairs, would be equally unreasonable. An examination of the hairs of the new mammal recently discovered by Sir Harry Johnston, K.C.B., pointed to the conclusion that the animal belonged to the genus *Equus*. The history of this discovery is now well known. Pieces of skin were first obtained, but not a complete skin, nor had the animal been seen alive. There was, however, other evidence besides that derived from the shape and structure of the hairs that the animal was equine. The skin was striped in a manner very suggestive of a zebra. On the strength of the evidence, Dr Sclater named the animal *Equus Johnstoni*. The resemblance between these hairs and the shoulder hairs in the Equidæ was shown by Dr Ridewood at a meeting of the Zoological Society,* and I myself can testify that whereas they do not resemble the hairs of any particular species of zebra especially, they do not differ more from the hairs of any such species than the species of zebras in their hair characters differ from one another.†

* P. Z. S., 1901.

† The Okapi's hairs, which I examined, are from a bandolier made from the skin from one of the legs of the animal (*vide* Sclater, P.Z.S., 1901). They

Sir Harry Johnston has more recently obtained a complete skin and two skulls of the animal, and these show that it is related to the extinct *Helladotherium* and may perhaps be referred to that genus.* Dr Ridewood, at a more recent meeting of the Zoological Society, exhibited microscopic preparations of the hairs of this animal and also of giraffe and antelope hairs, and pointed out that the hairs of the so-called *Equus Johnstoni*, while they differed from those of antelopes, resembled those of the giraffe and also those of the zebra.

The genus *Equus* contains some ten or more species, including two species of horses, three or four of asses, and a doubtful number of species of zebras. Three species of zebras are, however, well defined, namely, the Burchell's zebra (*Equus Burchelli*), the common or mountain zebra (*Equus zebra*) and the Somali or Grevy's zebra (*Equus Grevyi*), the skins of which are figured in Plates I., II. and III. Some account will now be given of the hairs of these zebras, after which the hairs of the horse will be referred to, and the paper will be concluded by a description of the hairs of certain zebra-horse hybrids and a reference to the telegony hypothesis.

Equus Burchelli.

In this, as in other zebras, the hairs are generally of stouter form than in the horse or ass, and the medulla in the case of the shoulder hairs at any rate is relatively thicker in the former than in the latter. The exact measurements for a typical hair from the shoulder region of the Burchell's zebra are as follows:—

Breadth of cortex on one side of medulla in three places.	Total breadth of hair in three places.
(1) ·018 mm.	·099 mm.
(2) ·018 mm.	·189 mm.
(3) ·027 mm.	·072 mm.

are about 5 mm. in length, or about the length of the shoulder and side hairs in the Somali and Penrice's zebra, from both of which they differ in shape, tapering to a point much more gradually. In the relative development of the medulla and cortex they closely resemble equine hairs, differing entirely from the hairs of antelopes, goats, and deer.

* Since the above was written Professor Lankester has named this animal, which is called the Okapi, *Ocapia Johnstoni*, Dr Sclater having already supplied the specific name.

(2) is taken in the middle of the hair-shaft, half-way between the point and the root; (1) is taken half-way between the point and (2); and (3) is taken half-way between the root and (2). A longitudinal streak, formed by pigment darker coloured than that colouring the rest of the cortex, may not infrequently be observed, so that the hair may appear, if mounted in a suitable position, differently coloured on one side of the cortex to what it is on the other. The medulla in many hairs is broken down in places and may be absent from the root for a distance as much as a quarter the length of the hair. The latter may reach 20 mm. Sections through the hairs of the mane which is upright are circular. Fig. 7 represents such a section. The line of demarcation between the cortex and medulla is irregular. The pigment is seen to be distributed much more thickly in that part of the cortex nearest the medulla than towards the periphery of the hair. The hairs here described are those of the Chapman's variety of the *Equus Burchelli.* This animal is regarded by Nathusius as a distinct species, as is also *Equus Böhmi.*

Equus quagga.

This animal, though undoubtedly a member of the Burchell's group of zebras, is commonly regarded as a distinct species. The hair characters are closely similar to those of the Chapman's zebra, but those of the side of the body tend to be longer and may reach 25 mm. in length; that is longer than the same hairs in any of the other zebras.

Equus zebra.

The shoulder hairs of the common or mountain zebra are not strikingly different to those of the Burchell's. The length is about the same. The following are measurements taken as with the Burchell's zebra hairs of the breadth of a typical shoulder hair and of the breadth of the cortex on one side of the same hair:—

Breadth of cortex on one side of medulla in three places.	Breadth of hair in three places.
(1) ·0144 mm.	·081 mm.
(2) ·0162 mm.	·090 mm.
(3) ·0162 mm.	·063 mm.

A longitudinal streak is commonly very distinct, and is often brought about by the presence of pigment on one side of the hair but not on the other. The medulla is wanting in the tip and root regions as in the Burchell zebra hairs. The hairs, including those of the mane, undergo a marked flattening. This is remarkable, seeing that sections through mane hairs, not only of the Burchell's but also of the Somali zebra, are nearly, if not quite circular, even those through the hair in the middle of its length where the degree of flattening is often greatest. The sections also show that the pigment is not specially aggregated towards the medulla, but is spread fairly evenly through the cortex, except in cases where the hair is longitudinally striped by pigment being present in much greater quantity on one side of the medulla than on the other. The line of demarcation between the cortex and medulla is parallel to the surface of the hair and not irregular as in the Burchell's zebra. Fig. 8 represents a section through a mane hair from a common zebra.

Equus Grevyi.

A study of the hairs of the Somali or Grevy's zebra leads to the conclusion that this zebra stands apart from all the others. Nathusius has commented on the extreme shortness of the hairs of the side of the body, their average length being about 5 mm. The breadths of the hair and of the cortex on one side, taken as before in three places, are as follows:—

Breadth of cortex on one side of medulla in three places.	Breadth of hair in three places.
(1) ·0162 mm.	·108 mm.
(2) ·0162 mm.	·162 mm.
(3) ·0216 mm.	·063 mm.

These measurements show considerable divergence from those of the other zebra hairs, and what is more, they are remarkably constant, being approximately the same for any fully grown hair drawn from the side of the body. The medulla is absent for some distance from the root, and where it makes its appearance is accompanied by a sudden thickening of the hair. Thus the hairs have long medullaless stalks. The pigment, which is thick in the greater part of the hair's length, becomes thinner passing along the

stalk, until near the root it is almost completely absent. It is apparently disintegrated in various places in the hair shaft. The most obvious character of these hairs is their remarkably short and stout form, being, relative to their length, much thicker than those of the other zebras, but actually very slightly thinner than those of the Burchell's zebra, that is, taking the measurements in the thickest part of the hair in each case. Sections through the hairs of the mane, like those of the Burchell's zebra, are circular almost throughout. The line of demarcation between the cortex and medulla is also almost regularly circular. The pigment is seen to be distributed pretty equally throughout the cortex, but has a slight tendency in places to be thicker nearest to the medulla and thinner towards the cuticle. This tendency was not apparent in the section from which fig. 6 was drawn.

The extreme shortness of the hairs on the side of the body can hardly be ascribed to want of vigour caused by the environment in which this zebra lives, for, as Nathusius points out, the hairs of the Somali ass, which lives under the same climatic conditions, are longer and better developed than those of any of the other wild asses.

Equus caballus.

The hairs of the horse, as might naturally be expected in a domesticated animal of which there are very numerous breeds, show extreme variability, so that it is practically impossible to state any characters which are applicable to all varieties of the species. The section figured, which is through a mane hair of Professor Ewart's "Circus Girl," the foal of a skewbald Iceland pony by a Shetland pony, is fairly typical. The characters there seen, such as the fine granular appearance of the evenly distributed pigment, the clear and regular line of demarcation between cortex and medulla, and the relatively narrow cortical region, are very common in transverse sections of mane hairs of horses. The shoulder hairs, speaking broadly, show a weaker development of the medulla and a thicker cortex than in any zebra hairs. Of course the length, breadth, and fineness of horse hairs are especially variable and depend largely on the breed.

Nathusius, who has but briefly described zebra hairs, has devoted

considerable space to horse hairs, referring to the characteristics of some of the breeds, so it is unnecessary to say anything on this subject here. Reference must, however, be made to a character upon which Nathusius, in his earlier papers at any rate, appears to lay considerable stress. I refer to the longitudinal striping so common in zebra and ass hairs. For some time he regarded this character as absent in horse hairs. Subsequently, however, he discovered longitudinal striping in hairs of certain ponies of mixed breeds imported from Russia. Although I have never observed such longitudinal striation in horse hairs, I know of no reason why it should not sometimes occur, especially in view of the fact that there is considerable evidence, as Professor Ewart * has shown, that the horse is descended from a striped zebra-like ancestor, and that this longitudinal striation is quite as well marked in the hairs of the asses, which are often supposed to have branched off from the ancestral equine stock, before the body striping was acquired in the Equidæ. It must, however, be doubtful how much stress should be laid upon such a character as variation in the degree of blending and arrangement of pigment, seeing that pigment in the other groups of the animal kingdom is known to be especially variable and easily influenced by the environment.

The following are measurements, taken as before, of a typical shoulder hair from a bay Irish mare :—

Breadth of cortex on one side of medulla in three places.	Breadth of hair in three places.
(1) ·027 mm.	·054 mm.
(2) ·027 mm.	·072 mm.
(3) ·027 mm.	·063 mm.

Asses.

Ass hairs are very fully dealt with by Nathusius in the two papers already quoted. It need only be mentioned here that longitudinal striping is very common in the shoulder hairs, and is sometimes seen also in those of the mane, and that the hairs show a marked degree of flattening, especially those of the Somali ass.

* Ewart, *The Penycuik Experiments.* London, 1899.

*Zebra-Horse Hybrids.**

The hairs of several of Professor Ewart's zebra-horse hybrids have been examined and sections cut. Seeing that the dams of these animals belong to different breeds, it might at first be expected that we should find quite as much diversity in the character of the hybrid hairs as in those of the dams. Such, however, is not the case, for the hairs of the hybrids are for the most part constant in shape and in the relative development of the medulla and cortex. The measurements, taken as before, of the shoulder hairs of the hybrid "Norette," whose dam was a Shetland pony, are not widely different from those of the sire, the Burchell's zebra :—

Breadth of cortex on one side of medulla in three places.	Breadth of hair in three places.
(1) ·0144 mm.	·081 mm.
(2) ·0162 mm.	·126 mm.
(3) ·0198 mm.	·054 mm.

They point to the conclusion that in the transmission of the character of the hair the Burchell's zebra is prepotent over the horse.

In some cases, however, the hybrid hairs do not resemble those of the sire any more than those of the dam, but this is not because they depart from the hybrid type, but because the hairs of the dam happen to be not dissimilar to zebra hairs. It has been mentioned that horse hairs, owing to the large number of breeds of horses, are very variable, and so it is not to be wondered at that in some cases sections through horse hairs should resemble sections through zebra hairs. This is the case with sections taken through the hairs of the mane of Professor Ewart's Clydesdale mare, "Lady Douglas," the mane hairs of whose hybrid offspring "Brenda" are if anything more like those of the dam than those of the sire, "Matopo." A more typical case is that of the hybrid "Sir John" (Plate IV.). Here the dam was a skewbald Iceland pony, "Tundra," and the sire the Burchell's zebra. Sections through the mane hairs of "Tundra" are identical in appearance with sections through hairs of "Circus Girl," which are figured.

* *Vide* Ewart, *The Penycuik Experiments*, London, 1899; and *Guide to Zebra-Hybrids*, Edinburgh, 1900.

Sections through the hairs of the hybrid offspring "Sir John," on the other hand, are in no way suggestive of those from the dam, but closely resemble those of the hybrids "Black Agnes" and "Brenda," one of which is figured (fig. 10). Professor Ewart has given reasons for the conclusion that of the existing species of zebras the Somali zebra approaches nearest to the ancestral type. He has also shown that the markings of the hybrids resemble the markings of the Somali zebra much more closely than those of the Burchell's zebra, and this resemblance he has ascribed to reversion. Now it cannot be said that the shoulder hairs of the hybrids, either in their shape, length, which is rather variable, or in the arrangement of the pigment, are at all suggestive of the same hairs in the Somali zebra. When, however, we compare the hairs of the mane the case is quite different. A section through a hair of the mane of a hybrid, such as the one figured, which is through such a hair in "Brenda," which in the mane hair characters is quite typical of the hybrids, shows a fairly even distribution of pigment and a circular line of demarcation between cortex and medulla, which are also what we find in a mane hair section from the Somali zebra. There is very little of that tendency of the pigment to become more thickly distributed towards the interior of the cortex, such as I have found in all sections through mane hairs of the Burchell's zebra. This is a curious result, and may, perhaps, like the peculiarities of the striping, be ascribed to reversion to the more ancestral type.

The Telegony Hypothesis.

Nathusius suggested that if the telegony hypothesis, or the hypothesis that subsequent offspring are infected by a previous sire be correct, we might expect to find evidence of it in the character of the hairs of the subsequent offspring. We have such a subsequent offspring in Professor Ewart's "Circus Girl." In 1897 the dam "Tundra" gave birth to a hybrid, "Hecla." In 1898 the subsequent foal "Circus Girl" was born, the sire being a bay Shetland pony. Just as "Circus Girl," both in make and colour, closely resembles her mother, so the hairs of the two animals are almost identical in character, and sections through the hairs of the manes are quite indistinguishable. There is nothing whatever suggestive

of the Burchell's zebra "Matopo," which was the previous sire. The same remark is equally applicable, so far as I have seen, to the other subsequent foals in respect of their hair characters.

I must express my indebtedness to Professor Ewart for providing the greater part of the material used, for kindly allowing me the use of the blocks from which Plates I.–IV. are reproduced, and for assistance in various other ways. To Mr Beddard, Prosector of the Zoological Society, I am indebted for what other material has been employed. In conclusion, I have great pleasure in thanking Sir Thomas Gibson Carmichael, Bart., for his very generous support.

Postscript, July 31st.—Since writing the above, Professor Ewart has been good enough to obtain for me, through the kindness of Mr Oldfield Thomas, some mane and shoulder hairs from a zebra skin recently brought home from Angola by Mr W. Penrice. Mr Thomas* describes the skin as possessing "the deeper and more essential characters of *Equus zebra*, such as the forward slope of the median dorsal hairs, the presence of a 'gridiron pattern' on the rump," etc., but differing from it "so much in other details that it clearly cannot be assigned to the typical form of that species." Mr Thomas adds that since it is isolated geographically from *E. zebra*, which is only known from South Africa, and differs from it in so many respects, in the absence of evidence of the existence of intermediate forms, it must be regarded as a distinct species, which he calls *Equus Penricei*.

The characters of this animal's skin are briefly described by Mr Thomas.* I find that the individual hairs from the region of the shoulder resemble closely those of the Somali zebra (*E. Grevyi*), which, according to Mr Thomas, Penrice's zebra also resembles in "the equal striping of the body, the short close fur, and the buffy tone of the light stripes." The following are measurements of a

* Oldfield Thomas, "On *Equus Penricei*, a Representative of the Mountain Zebra discovered by Mr W. Penrice in Angola," *Annals and Mag. of Nat. Hist.*, vol. vi., November 1900.

typical shoulder hair, taken in three places, as with the other specimens of hairs described in this paper :—

	Breadth of cortex on one side of medulla in three places.	Breadth of hair in three places.
(1)	·0144 mm.	·126 mm.
(2)	·0162 mm.	·153 mm.
(3)	·027 mm.	·09 mm.

In length the shoulder hairs are scarcely more than those of the Somali zebra, being usually a little over 5 mm. They are appreciably flattened in the middle, agreeing in this respect with most equine hairs. A longitudinal striping can be observed in some of the hairs. The medulla is not present for a considerable distance from the root, and where it arises the hair thickens out rapidly as with shoulder hairs from the Somali zebra. On the other hand, the medulla extends almost to the hair's tip. Pigment of a lighter colour than that of the rest of the hair is present throughout the medullaless region in the coloured hairs. Transverse sections through the hairs of the mane present a circular outline. The pigment in the cortical region is evenly distributed between the cuticle on the one side and the medulla on the other. The line of demarcation between the cortex and medulla is extremely irregular instead of being parallel to the cuticle.

DESCRIPTION OF THE PLATES.

Plate I.

Fig. 1. Skin of Burchell's zebra (Crawshay's variety).

Fig. 2. Skin of mountain or common zebra.

Plate II.

Fig. 3. Skin of young Burchell's zebra from British East Africa.

Plate III.

Fig. 4. Skin of young Somali zebra.

Plate IV.

Fig. 5. " Sir John," a hybrid between a Burchell's zebra and a skewbald Iceland pony.

Plate V.

Fig. 6. Transverse section through a mane hair of a Somali zebra, showing the fairly regular line of demarcation between the cortex and medulla, and an equal distribution of pigment throughout the cortex. × 300 diam.

Fig. 7. Transverse section through a mane hair of a Burchell's zebra, showing the irregular line of demarcation between the cortex and medulla, and the much thicker distribution of pigment towards the interior of the cortex. × 300 diam.

Plate VI.

Fig. 8. Section through mane hair of a common zebra. × 300 diam. (The section from which this is drawn is not quite transverse, this being indicated by the appearance of the pigment. The fact that the long axes of the lines of pigment—which are, as usual, arranged longitudinally—do not lie in the same direction as the long axis of the section, proves that an absolutely transverse section is not circular. This is completely borne out by the appearance of other sections through the mane hairs of this zebra.)

Fig. 9. Transverse section through a mane hair of the pony "Circus Girl," showing an almost regularly circular line of demarcation between medulla and cortex and finely granular pigment which is equally distributed throughout the cortical layer. × 300 diam.

Fig. 10. Transverse section through a mane hair from the zebra-horse hybrid "Brenda," showing general resemblance to fig. 6. × 300 diam.

Figs. 1 and 2 are from *The Penycuik Experiments*, Ewart.

Figs. 3, 4 and 5 are from *Guide to Zebra Hybrids, etc.*, Ewart.

Figs. 6–10 were drawn by Mr Richard Muir from sections passing in each case approximately through the middle of the hair's length. The cuticular portion is represented in all the figures by the narrow unpigmented layer outside the cortex.

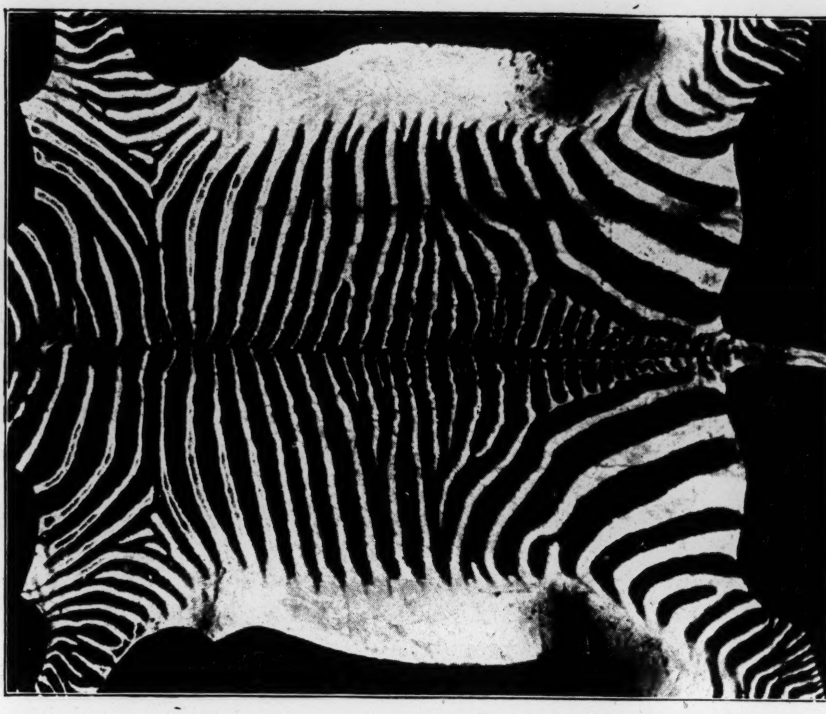


Fig. 2.

Fig. 1.

F. H. A. Marshall.—Plate I.

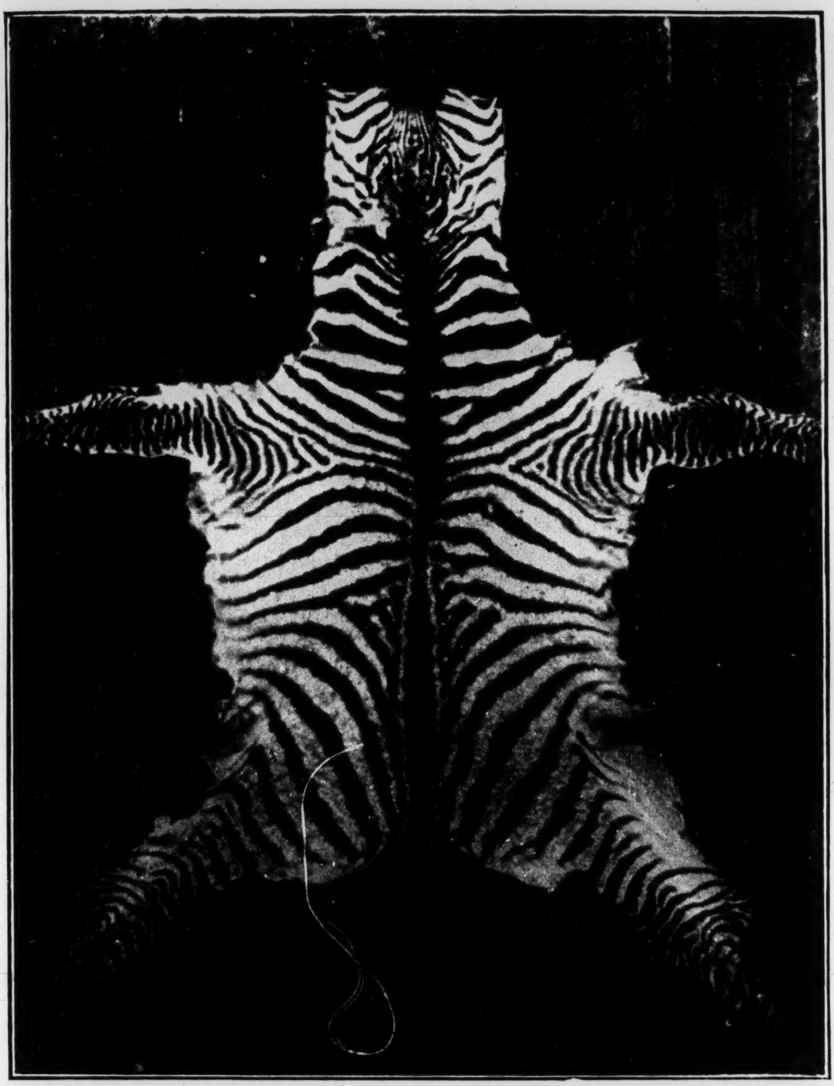


Fig. 3.

F. H. A. Marshall.—Plate II.

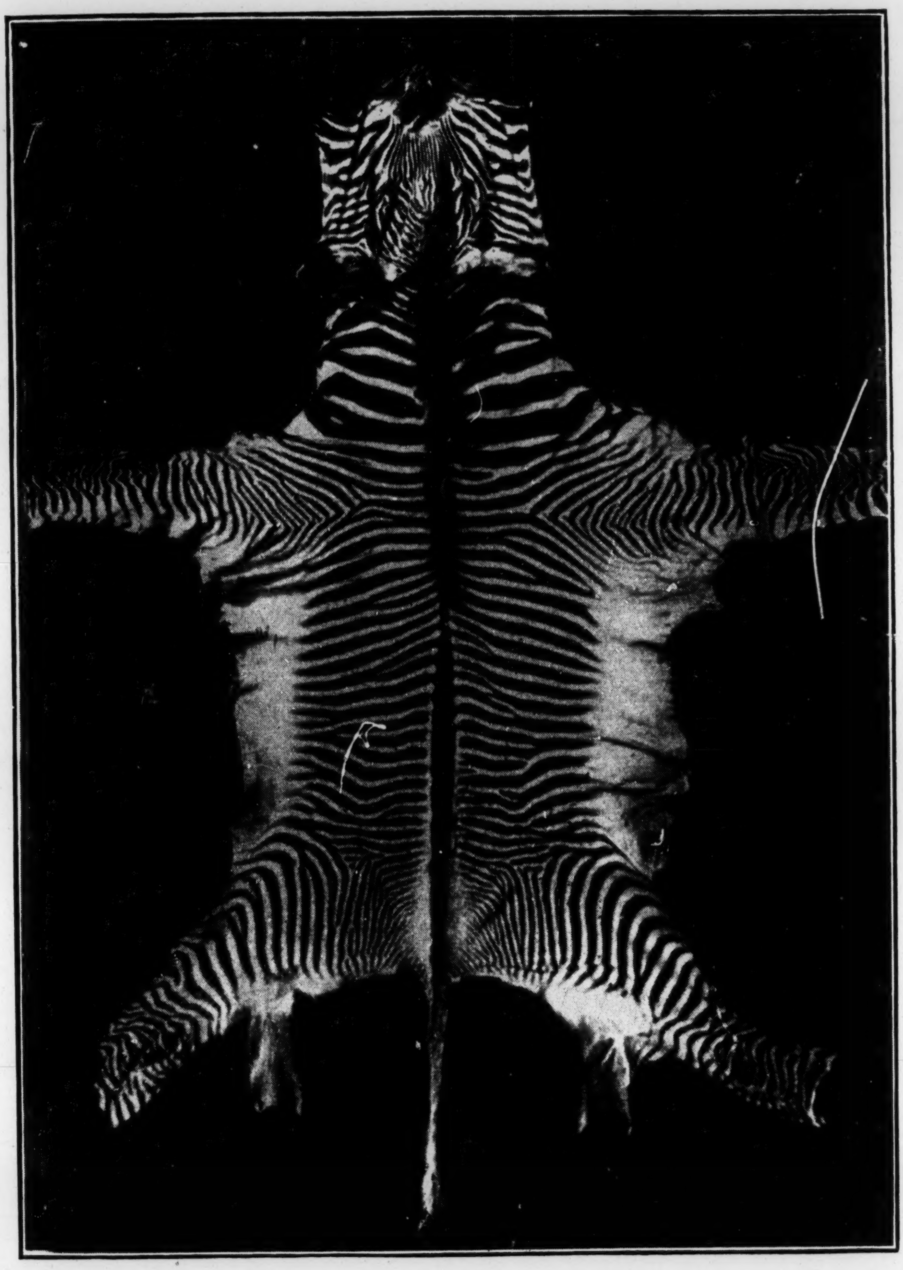


Fig. 4.

F. H. A. Marshall.—Plate III.

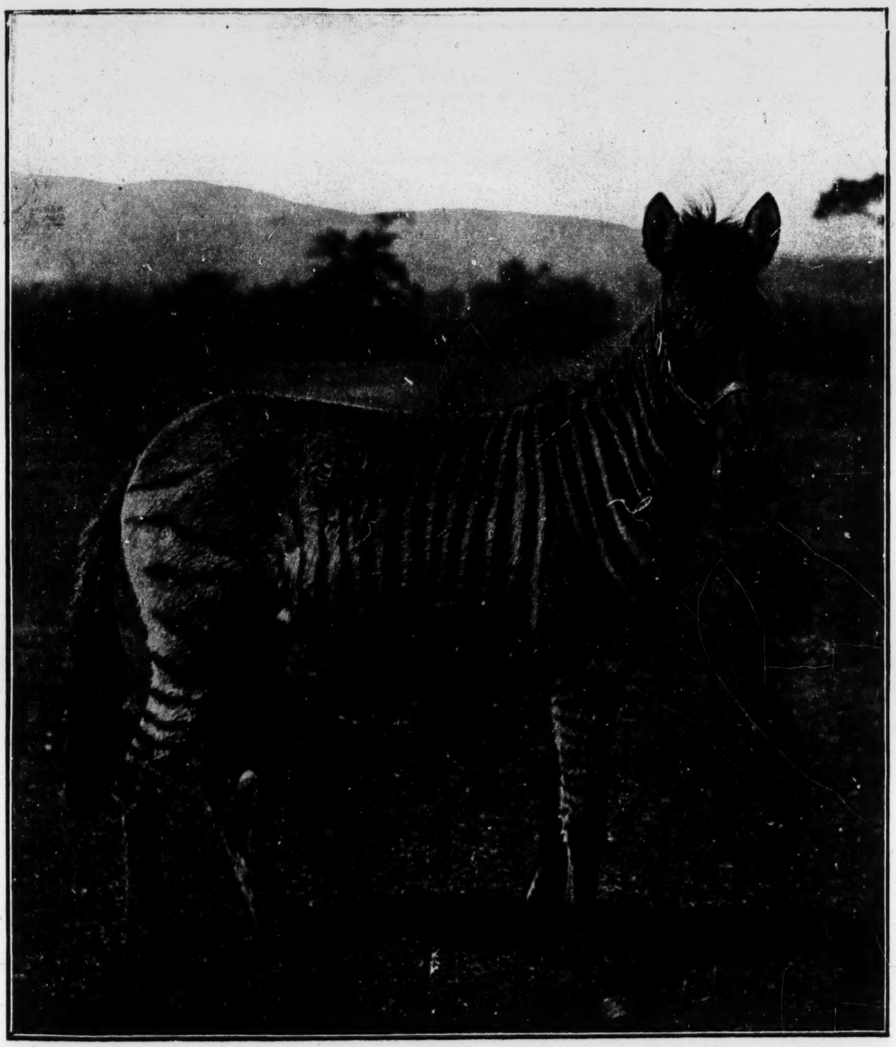


Fig. 5.

F. H. A. Marshall.—Plate IV.

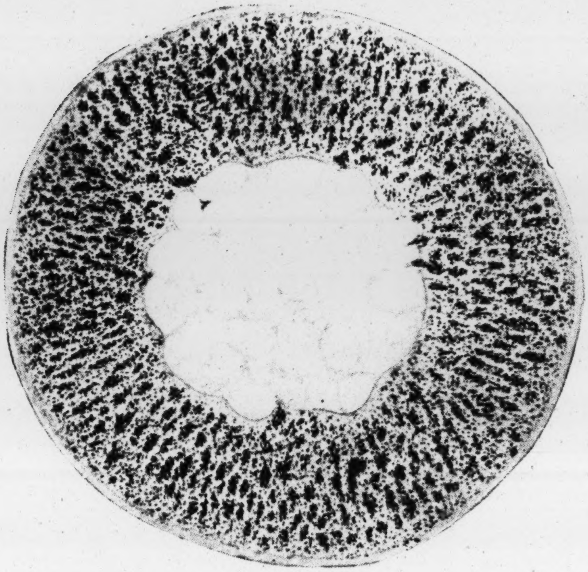


Fig. 6.

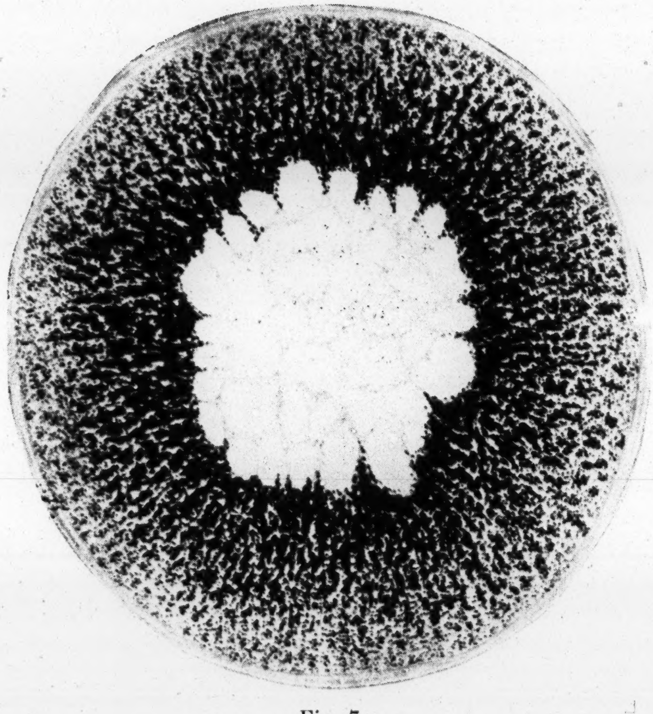


Fig. 7.

F. H. A. Marshall.—Plate V.

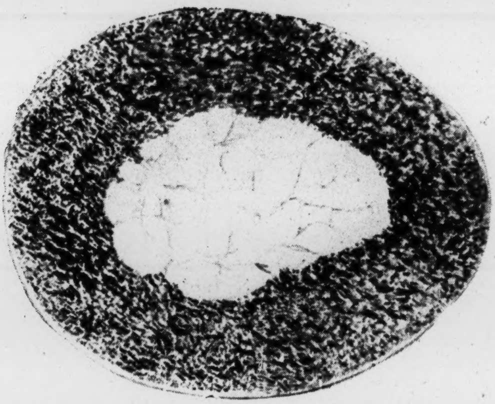


Fig. 8.

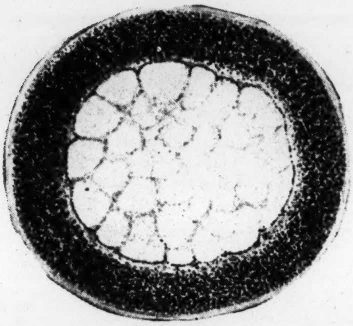


Fig. 9.

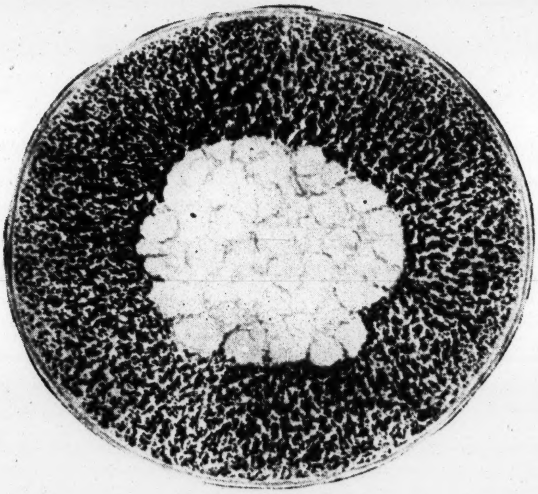


Fig. 10.

F. H. A. Marshall.—Plate VI.

Notes on the Appearance of some Foraminifera in the Living Condition, from the 'Challenger' Collection. By Frederick Chapman, A.L.S., F.R.M.S. *Communicated by* Sir John Murray, K.C.B., F.R.S. (With Three Plates.)

(Read July 15, 1901.)

The habits and mode of existence of Foraminifera are always interesting subjects to students of the Protozoa, and this fact alone might perhaps justify the following notes, even were they not accompanied by the valuable drawings prepared by Mr G. West, from pencil sketches and microscopic slides made by Sir John Murray from the living Foraminifera collected during the voyage of H.M.S. 'Challenger.'

The writer is greatly indebted for the privilege of examining and describing these drawings, and a collection of mounted specimens of a like character, to Sir John Murray, K.C.B., LL.D., F.R.S., who generously placed them in his hands a year or two ago.

The species of Foraminifera depicted on these plates are :—

Textularia conica, d'Orbigny.
? Discorbina globularis (d'Orbigny).
Truncatulina lobatula (Walker and Jacob).
Anomalina polymorpha, Costa.
Carpenteria balaniformis, Gray (young specimens).
Pulvinulina elegans (d'Orbigny) [the deep-water variety, *P. Partschiana* (d'Orbigny)], and
Amphistegina Lessonii, d'Orbigny.

Plate I.

The examples of living Foraminifera shown on this plate were obtained from two stations in the Pacific—No. 192a (Sept. 26, 1874); lat. 5° 49′ 15″ S., long. 132° 14′ 15″ E. Off Ki Islands, Banda Sea. Depth 129 fathoms. Sandy mud (H. B. Brady).

Also No. 232 (May 12, 1875); lat. 35° 11′ N., long. 139° 28′ E. S. of Japan (Hyalonema ground). Depth 345 fathoms; bottom

temperature 41·1° F., surface temperature 64·2° F. Green mud (Murray and Renard).

The central figure on Plate I. is that of a fine specimen of *Textularia conica* (fig. 1). The test is rather larger than usual, consisting of no less than twenty-five chambers; the initial series being practically hyaline or sub-arenaceous in structure. This example is seen to be creeping along a smooth spicule of *Hyalonema*, with the granular sarcode completely covering the oral surface of the test. There is no sarcode emission, apparently, from the lateral surfaces of the test, and this would point to its imperforate character. From Station 232, S. of Japan, 345 fathoms.

Figs. 2 and 3 are typical specimens of *Truncatulina lobatula*, fig. 2 showing the superior, and fig. 3 the inferior surface of the shell. The protruded sarcode in these examples seems to form somewhat ragged extensions, which partially separate from the main mass surrounding the oral opening of the shell, and are probably emitted from the tubules, forming by themselves a knotted reticulum. These specimens were found moving over the surfaces of various marine algæ. Station 232, S. of Japan, 345 fathoms.

The remaining figures, 4, 5, and 6, on this plate, are examples of the curiously variable and interesting species *Anomalina polymorpha*. In this form we have a remarkable instance of the adaptability of the foraminiferal shell to the surfaces over which the organism moves. This species presents two modifications, one with longish, round-ended spines, and the other, not so frequent, without processes. The latter form resembles *Discorbina rugosa* very closely, but is as a rule never so regularly shaped; and it is, moreover, always associated with the spinous variety.* The specimens shown in figs. 4 and 5 were found attached to marine algæ, and, it will be remarked, are fairly regular in the coiling of the shell. The other specimen, shown in fig. 6, has adapted its shell to the form of the object of attachment, the spicule of *Hyalonema*; and the coiled shell, besides being laterally elongated, is hollowed along the longer axis, on its inferior surface, so as to be more securely seated on the sponge-spicule. There is little doubt that these modifications of *Anoma-*

* See H. B. Brady's remarks, *Rep. Chall.*, vol. ix., 1884, p. 676.

lina polymorpha could easily move along the rod-like spicule when living; and in that condition always appear to have carried an arming of slender sponge-spicules round the region of the oral aperture, which might serve to guide the extruded sarcode and act as axes of support.

In this remarkable adaptation of a foraminiferal shell to the surface on which it lives, *Anomalina polymorpha* shows a parallelism with *Orbitolites marginalis*, which at Funafuti was found to frequently present the most unconventional modifications of the ordinary discoid form, often appearing as a sinuous, contorted or S-shaped series of chamberlets when seen in vertical section in the cores of the Atoll-boring; and in the lagoon* it was often found to have attached itself to the fronds of *Halimeda*, and even to have wound itself round the cylindrical stems. Both in the case of *Anomalina* and *Orbitolites*, the more regular form seems to be the simpler in construction, because formed on a uniform and successional plan of growth, the wild-growing varieties being a later and hence secondary modification. In the examples quoted, it is possible that this anomalous *Anomalina* was derived from the regular *Discorbinæ*, and from *Orbitolites* the genus *Nubecularia* may have been derived through the more regular or intermediate genus *Miliolina*.

Figs. 4 and 5 represent specimens from Station 232, and fig. 6 from Station 192A.

Plate II.

The specimens shown on this plate were obtained at Station 344, (April 3, 1876), off Ascension; lat. 7° 54′ 20″ S., long. 14° 28′ 20″ W.; depth 420 fathoms.

The specimens of Foraminifera represented in fig. 1 are probably the young of *Carpenteria balaniformis*, Gray. This species is in its earliest stage remarkably like the erect forms of the Rotaline type, as *Truncatulina refulgens* and *Pulvinulina Micheliniana*.† These young forms are seen living attached to the stems of hydroids, and a noteworthy feature is the presence of a conspicuous bunch of

* Chapman, "On Foraminifera from the Funafuti Lagoon," *Journ. Linn. Soc. Lond., Zool.*, vol. xxviii., 1901, p. 181, pl. xx. figs. 1–3.

† *Rep. Chall.*, vol. ix. p. 677.

sponge-spicules grouped round the mouth of each test. This habit of collecting sponge-spicules is common to the other species of *Carpenteria*, and in some cases, notably in *C. rhaphidodendron*, the sponge-spicules are often enclosed in the sarcode within the test in such abundance that at first sight the animal resembles a sponge rather than a foraminifer.

The other specimens on Plate II. (fig. 2) are a small variety of the deep-water type of *Pulvinulina elegans*, and these, similarly with the *Truncatulinæ* before mentioned, have an irregular mass of sarcode surrounding the test.

PLATE III.

The Foraminifera figured on this plate were obtained off St Vincent, in the Cape Verd Islands, at a depth of 10 fathoms (30th July 1873).

Amphistegina Lessonii, of which there are numerous specimens in this dredging, is here seen to be attached to an algæ, and its sarcode almost covers the weed in places. Another and much smaller species accompanies the *Amphisteginæ*, bearing a strong resemblance to *Discorbina globularis*; but this is not quite clear in the absence of specimens preserved in the mountings, which the writer has examined for them, but without success.

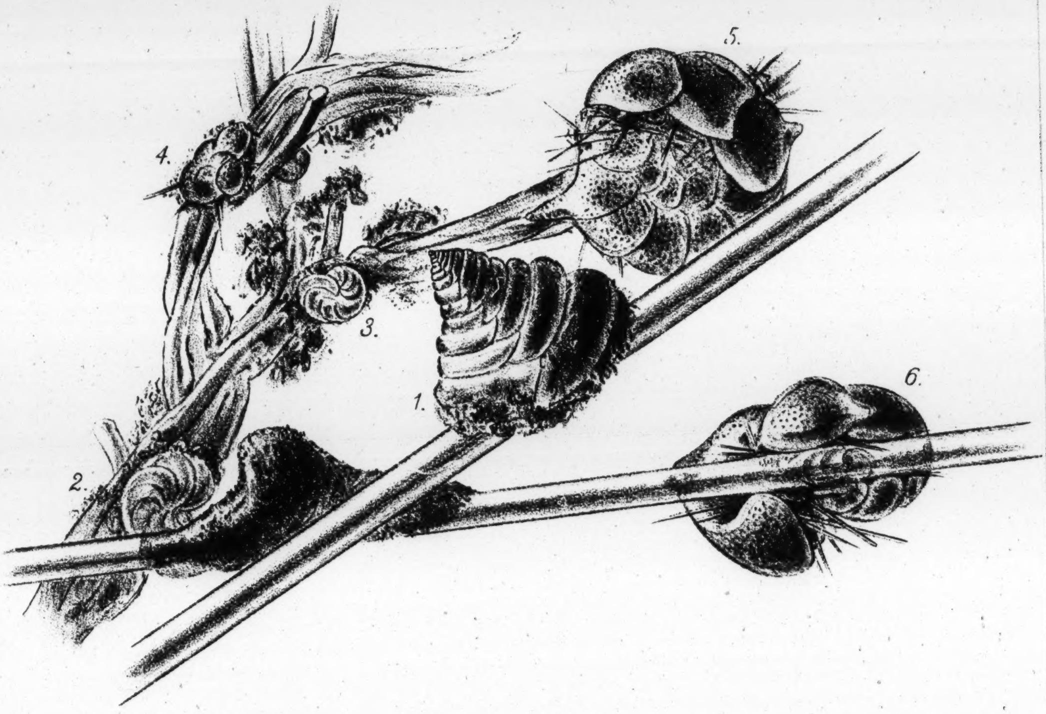
The species upon which the above remarks have been made, illustrated by the beautiful drawings by Mr West, will, the writer ventures to think, amply show the interest attaching to any records relating to the appearance and habits of living Foraminifera; and this may be an incentive to those who have opportunities for collecting and preserving these tiny creatures when they are in the living condition to add to our knowledge in this direction, and especially to note any facts regarding the changes or development of the animal during its life's history.

PLATE I.

Fig. 1. *Textularia conica*, d'Orb. Station 232, S. of Japan, 345 fathoms.

Figs. 2, 3. *Truncatulina lobatula* (W. & J.). Station 232, S. of Japan, 345 fathoms.

CHAPMAN: FORAMINIFERA, —— Plate I.



McFarlane & Erskine Lith. Edinr.

CHAPMAN: FORAMINIFERA,—– Plate II.

M'Farlane & Erskine, Lith. Edin.

CHAPMAN: FORAMINIFERA,— Plate III.

McFarlane & Erskine, Lith. Edin.

Figs. 4, 5, and 6. *Anomalina polymorpha*, Costa. 4 and 5 from Station 232, S. of Japan, 345 fathoms; 6 from Station 192A, off Ki Islands, 129 fathoms.

PLATE II.

Fig. 1. *Carpenteria balaniformis*, Gray (young specimens). Station 344, off Ascension, 420 fathoms.

Fig. 2. *Pulvinulina elegans* (d'Orb.). Station 344, off Ascension, 420 fathoms.

PLATE III.

Fig. 1. *Amphistegina Lessonii*, d'Orb. Off St Vincent, 10 fathoms.

Fig. 2. ? *Discorbina globularis* (d'Orb.). Off St Vincent, 10 fathoms.

Photographs of the Corona taken during the Total Solar Eclipse of May 28th, 1900. By Thos. Heath, B.A. (With Five Plates.)

(Read July 15, 1901.)

In June of last year I had the honour of reading before this Society a preliminary account of the Scottish expedition for the observation of the Total Solar Eclipse of May 28th, 1900, at Santa Pola, on the south-east coast of Spain (long. 0° 30′ W., lat. 38° 13′ N.). I have now to lay before the Society the results of the part of the expedition specially assigned to me, which was to obtain photographs of the Corona. I succeeded in securing four, that being the largest number I considered it advisable to attempt in the very short total phase of eclipse, only 75 seconds being available for the exposures and necessary manipulation of the camera backs.

According to my original plan, I had arranged to expose the four plates as follows:—The first immediately after totality commenced, with an exposure of 1 second; for the second I allowed an exposure of 6 seconds; for the third, 15 seconds; and for the fourth, 1 second. For each of the three intervals between successive exposures necessary for turning the backs, closing and opening the slides, etc., I found I had to allow 15 seconds.

I drilled myself for several days before the eclipse, till I found I could get through my programme quite comfortably in the time allotted to each part, and finish with my fourth plate exposed a few seconds before totality ended. In the agitation which is almost inseparable from the supreme moment of an eclipse, I suppose I must have made some of my intervals rather longer than I had arranged, with the result that my last plate appears to have been exposed at the critical moment when the sun was just beginning to reappear outside the western limb of the moon. This fact is well shown on the photograph. The presence of the sun has not, however, in any way interfered with the success of the

photograph as a picture of the Corona. I regret to say that, owing to the amount of light in the sky during the whole progress of the total phase, the two longer-exposed plates show more or less fogging of the background, making it rather difficult to obtain good prints. This is more especially the case with No. 3, which had the longest exposure.

The instrument with which the photographs were obtained is an equatorially mounted telescopic camera, belonging to the Royal Observatory, Edinburgh, with a Cooke triple object-glass of 6-inch aperture and 104 inches focal length. The object-glass had been only recently acquired by the Royal Observatory when the eclipse took place. It had, however, been mounted sufficiently long to allow of its being carefully tested by Professor Copeland, who concluded that it was admirably suited for such a purpose as photographing the Corona. A few trial photographs were also made for the purpose of determining the focus, and at the same time testing the photographic definition. Amongst others, the trail of the double star ζ Ursæ Majoris was photographed. On developing, the trail was found to be distinctly double in all its length. The difference of declination of the two components is $12''\cdot6$. If we compare this with the moon's angular diameter and her diameter measured on the eclipse plates $= 0\cdot94$ of an inch, we will find the distance between the two trails to be $\frac{1}{162}$ of an inch.

It will be seen from the photographs that the whole of the moon's disc is surrounded by coronal light, but that the rays about the sun's polar regions are very much shorter than those which emanate from the regions about the equator, the usual form of Corona at minimum of sun-spots.

The long streamers stretching out to the east and west occupy about 135° of the limb on each side, and are nearly symmetrically placed with reference to the sun's equator. The two sides, however, present quite different configurations, in their outermost extensions more especially. The western streamer has its longest extension at the sides, which reach outwards about a solar diameter and a half, as measured on photograph 3, the northern edge being somewhat longer than the southern. These edges start from the limb in beautifully curved lines for about half their length; the outer halves, on the contrary, are straight and

slightly divergent. The portions of this streamer lying inside the edges fade away more rapidly than the sides, giving it somewhat the appearance of a swallow-tail. On the eastern side there are four streamers, one of which is, however, much longer and more conspicuous than the others, and is of about the same length as the edges of the western. The extreme ends of these four rays can be easily seen separated; but, up to a distance of about half the sun's diameter from the limb, they coalesce, forming together the bright inner region of the Corona. From a careful study of the photographs, it seems to me that the eastern section of the Corona is made up of four roughly conical streamers, whose bases overlap one another to some extent at right angles to the line of sight. The western section, on the other hand, would appear to be composed of several streamers; three, at least, can be made out, whose bases do not overlap, or do so only to a small extent.

The polar regions present a great contrast to the east and west equatorial regions. They are much more contracted in extent along the limb, covering only about 45° at each pole, and instead of the long, far-reaching streamers, show only short feathery tufts, seven or eight in number at each pole. They curve away from the north and south poles of the sun's axis, and collectively give one the idea of groups of feathers arranged as plumes. There is no appearance on any of these photographs of the dark rifts which have been found on some other photographs of this eclipse, and of some previous eclipses, such as 1896. The Corona surrounds the limb at all points, and in the spaces between the tufts the light fades away to so small an extent, and so gradually, that it is in some cases difficult to be sure that there is a division between the rays, without very careful examination of the negatives.

As to the possibilities of the photographic method for giving large-scale pictures of the Corona, there is no doubt that it is only since the introduction of photography into the regular work of eclipse-observing that reliable pictures of the forms of the Corona have been obtained. Though several good photographs had previously been made by Dr De la Rue and others, it was not before the eclipses of 1870 and 1871 that really successful pictures of the outer regions of the Corona were obtained. In the latter year Mr Davis at Baikul and Mr Henessy at Dodabetta succeeded

in obtaining photographs which for beauty of detail have not since been much, if at all, improved upon. Reproductions of drawings made from the combined negatives of each of these observers will be found in vol. xli. of *Memoirs of the Royal Astronomical Society*. A glance at these two pictures will show their remarkable resemblance, and even careful examination fails to show differences between them in more than a few of the minuter details. In each the Corona extends to rather less than a solar diameter from the limb. As to the drawings made by hand from visual observations with telescopic assistance—and this is also true of every eclipse observed in this way—there is nothing so remarkable as their dissimilarity. On the other hand, Captain Tupman's drawing depicts the Corona extending to fully $1\frac{1}{2}$ diameters from the limb, as compared with less than one diameter of the photographs.

Somewhat similar has been the result of the 1900 eclipse. None of the photographs, or reproductions of photographs, which have come under my notice, show so far-reaching a Corona as is shown in what I quite believe is a most faithfully executed drawing. I refer to the drawing by Dr A. Wolfer of Zurich and two colleagues, published in the *Archives des Sciences Physiques et Naturelles* of Geneva. While in my photographs the Corona reaches outwards about a diameter and a half, Dr Wolfer's drawing shows it extending more than two diameters, and a very striking peculiarity of the drawing, as compared to the photographs, is that the eastern extension, instead of coming to a point, is spread out to a shape very similar to the western extension.

It would appear, therefore, that photography, as at present practised, has its limitations in the direction of coronal work, and by no means does away with the usefulness of trustworthy drawings. These limitations are particularly felt in such an eclipse as that of May 1900, on account of the short duration of totality and the general brightness of the sky. There seems no reason to suppose, however, that photographs of the Corona could not be taken in a long total eclipse, of say five minutes' duration, which would show the extensions as far out as they were visible to the eye, unless we are to adopt the suggestion which has been made, that the outermost regions are less rich in actinic light, as compared

with visible light, than the parts nearer to the sun. I think, however, that further efforts should be made with the most suitable instruments available, before photography has to confess itself unable to do for the whole of the Corona what it has already done for the greater part of it.

Plates I., II., III., and IV. have been reproduced from the photographs, and show fairly well the general appearance of the Corona, though the details are necessarily not so distinct as they are in the negatives.

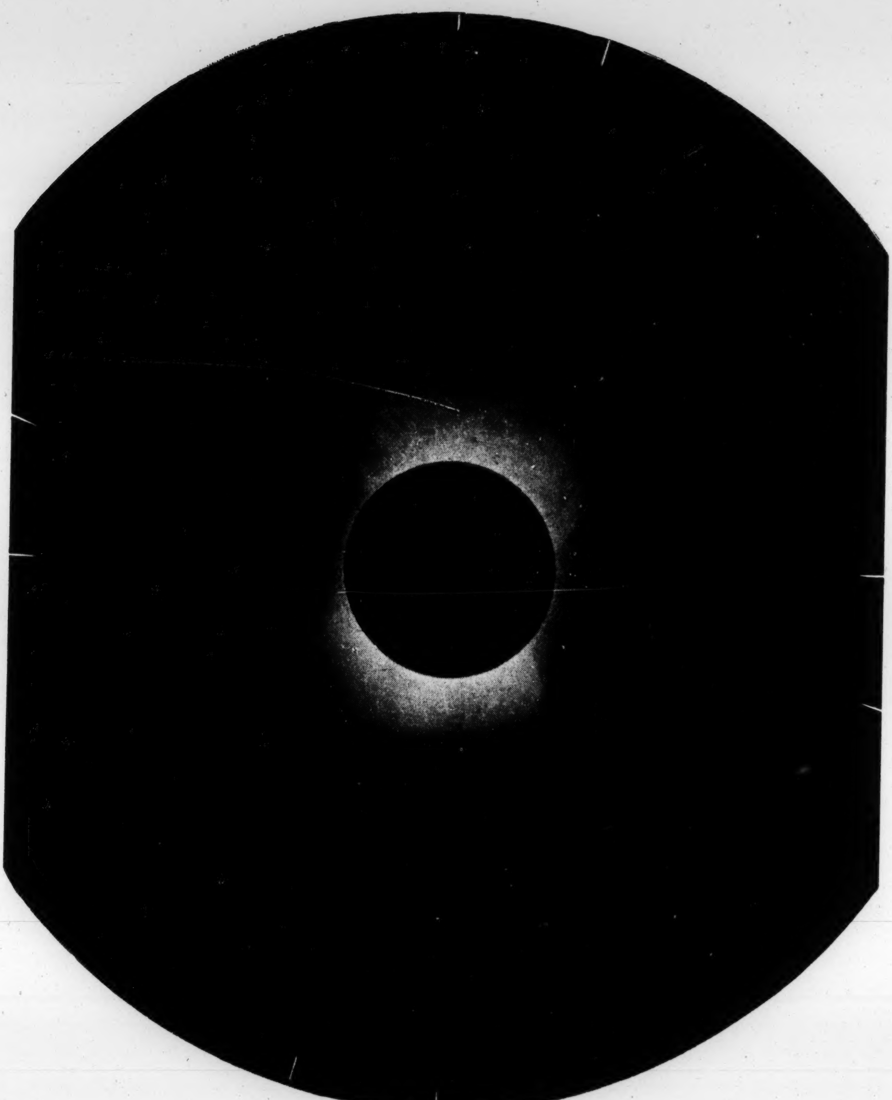
Plate V. is a reproduction of a drawing made from the negatives, and is intended to show the details of the Corona in stronger contrast than they present in the originals. Proofs of this drawing have been compared with the negatives, and no detail has been discovered which is not found on more than one of them, with the exception of a very faint wisp of light which appears to emanate obliquely from the south edge of the great west streamer. This is to be seen only on the longest exposed negative. The outmost extensions of the Corona have also been drawn, as shown in this negative.

W.

Sun's Equator.

Sun's Axis.

N.



S.

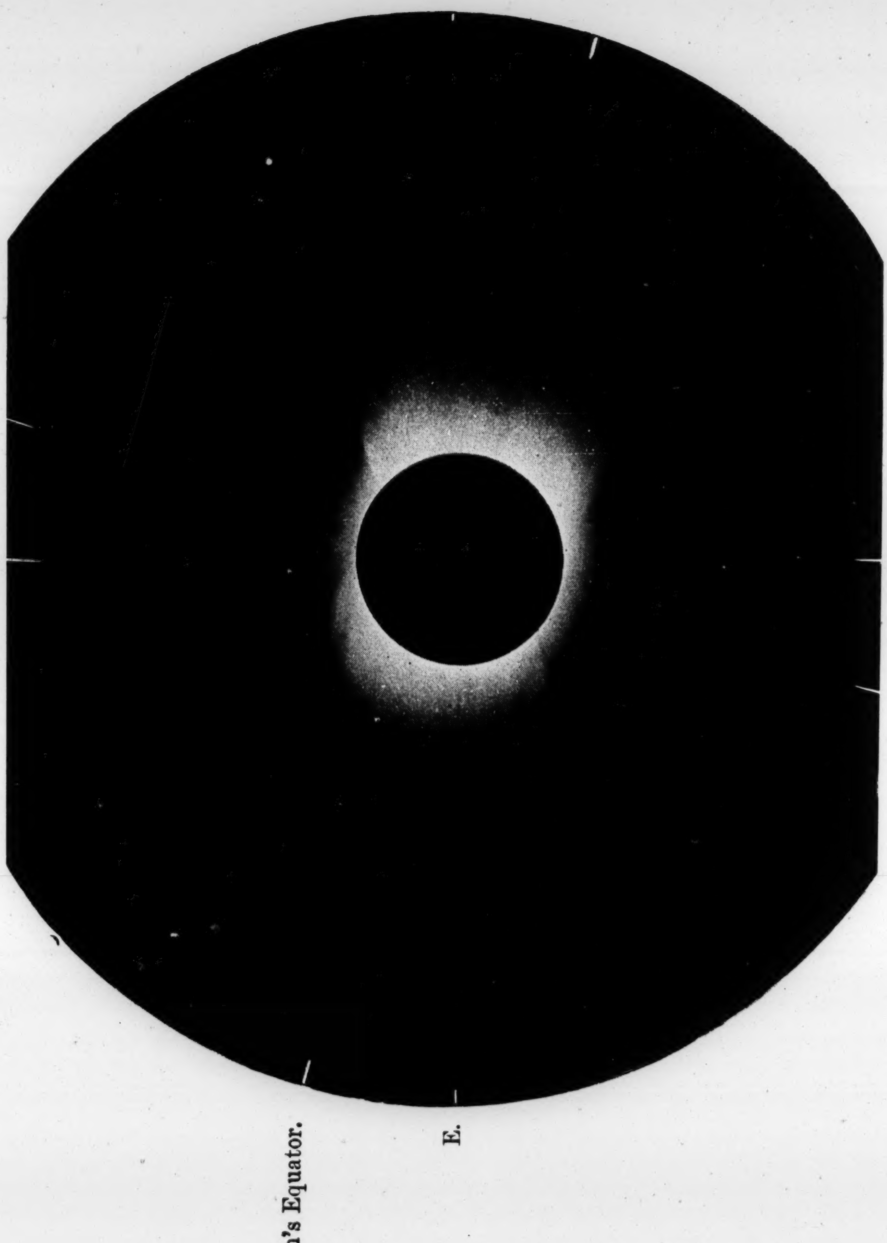
Sun's Axis.

Sun's Equator.

E.

T. Heath.—Plate I.

T. Heath.—Plate II.



T. Heath.—Plate III.

W.

Sun's Equator.

Sun's Axis.

N.

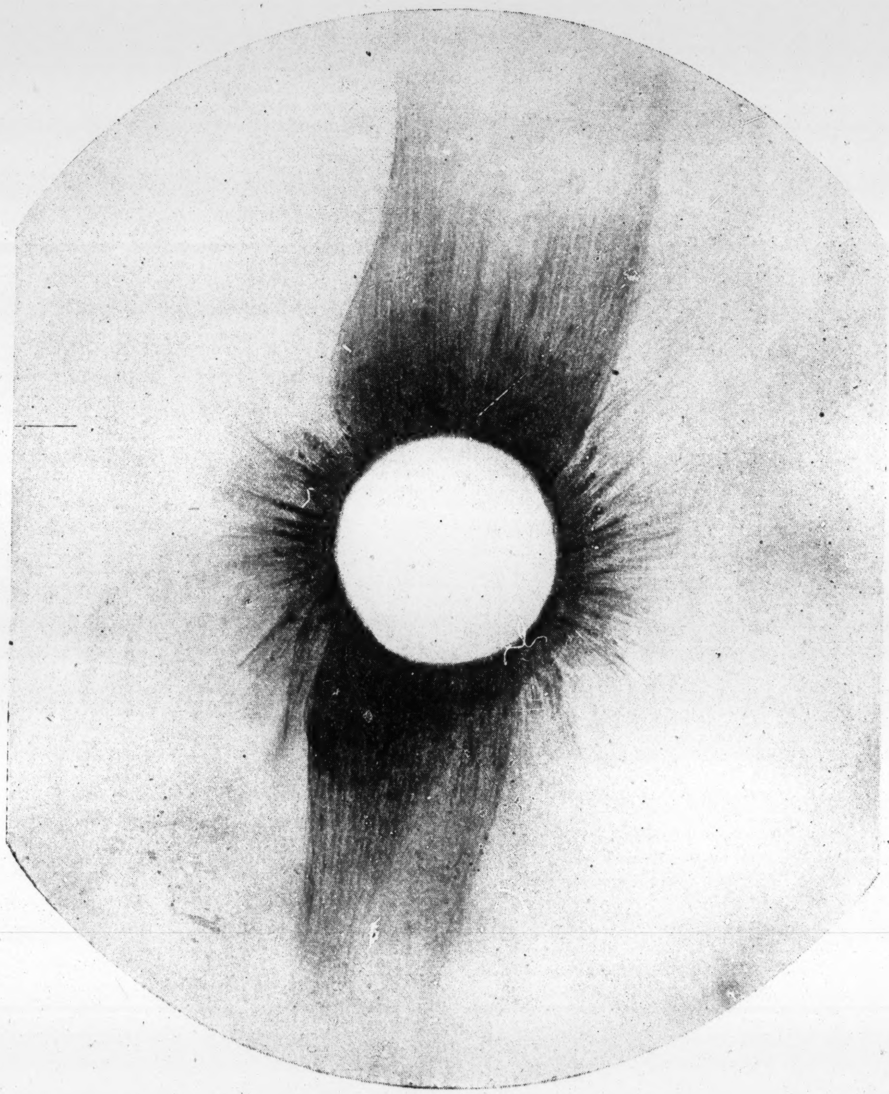
S.

Sun's Axis.

E.

Sun's Equator.

T. HEATH.—Plate IV.



A. H. H. Delt.

T. HEATH.—Plate V.

Observations on Binary Fission in the Life-History of Ciliata. By Dr J. Y. Simpson. (With Two Plates.)

(Read June 3, 1901.)

The simplest and most common form of reproduction amongst the Ciliata is binary fission. In this ordinary, possibly vegetative, method of reproduction the plane of division is generally perpendicular to the long axis of the creature. To this generalisation the Vorticellidæ form an apparent exception, but on the view that their evident long axis really corresponds to the dorso-ventral axis of other ciliates their case falls into line with that of the rest of the sub-class. Formerly many instances of fission in the direction of the long axis were described; they may safely be considered to have been mere instances of conjugation.

Binary fission most commonly takes place while the creature moves about; *i.e.*, it is (in most cases at least) an activity temporarily added to all the other activities of ciliate existence. As such it may be considered to be the original method of reproduction. Under other conditions fission may take place when the creature is at rest; or, in other words, in certain cases binary fission is not associated with free movement; on the contrary, this stationary fission is usually associated with the formation of a cyst. Under these circumstances the operation may take place more than once in succession. Such stationary fission together with budding—which is simply a form of fission where the products are so unlike in size as to be distinguishable as parent and offspring—are best considered as modifications of ordinary binary fission.

By ordinary binary fission, then, we understand the division of a ciliate during its active free existence into two daughters by a constriction more or less transverse to its long axis. It is confessedly difficult to arrive at a rationale of binary fission. One might suppose that it was associated with a certain limit of size, and that, as is more evident in the case of globular Rhizopoda, since the bulk increases as the cube of the diameter while the surface in-

creases only as the square, relief is obtained by the process. But it has been shown more than once—indeed is matter of common observation—that binary fission takes place at all stages in the development of certain Infusoria, and is not merely postponed till they reach a definite size. That is to say, binary fission is not necessarily connected with growth beyond the specific mass of the species. For example, in encysted forms, there is no possibility of growth previous to division, and in other cases, as the result of continuous division, there may be an actual decrease in size. To put the matter briefly, actual increase in size is neither a constant precursor or result of binary fission *per se* in the case of the Infusoria. All that can be said is that while in certain cases, *e.g.*, *Stylonichia*, *Euplotes*, a distinct lengthening is noticeable at the commencement of the process, in others, *e.g.*, *Stentor*, *Spirostomum*, no such phenomenon is observable.

Another question has interest in this connection—where are the first signs of the process noticeable, in the nucleus or in the cytoplasm? In view of the fact that both answers have been given by first-class workers, Bütschli contents himself with stating for the majority that there are undoubtedly many instances where there are hints of new formations in the plasma, *e.g.*, the "anlage" of the new ciliary apparatus, mouth, or contractile vacuoles, before any change in either macronucleus or micronucleus is observable.

Simple binary fission, apart from nuclear considerations, is not a very complicated process. As already stated, the plane of division lies more or less at right angles to the long axis of the body, and usually approximately near the middle. It is necessary to make the qualification "more or less," because in the case of *Spirostomum teres*, at any rate, the plane of division is somewhat oblique, as Stein observed so long ago as 1867 (*Der Organismus der Infusionsthiere*, Bd. II.). In the more highly organised Ciliata the special organs have to be duplicated, and this is achieved either by division of the already existing organ or by fresh formation in one of the offspring. The former method is comparatively rare, and only occurs where the organ or system in question, as, *e.g.*, the canals connected with the contractile vacuoles, runs practically the entire length of the creature. When the twin

organ is entirely formed anew, the position of the original one usually decides upon which of the daughters this work devolves. Be it in the anterior portion of the parent, it is generally found that the daughter that develops from the posterior half will have to form the new organ. Of *Paramecium*, however, it may be said that the mouth lies as a rule slightly in the posterior half of the body; and in this particular instance the new mouth is always formed *behind* the old one, and this involves considerable consequent displacement. One or two abnormal *Paramecia*, in which the mouth lay unusually close to the tail, gave one the opportunity of verifying the correctness of this exception, even in such extreme cases. It is, however, the anterior contractile vacuole that is formed anew in either daughter.

In the case of the Hypotricha there is a renewal of the whole ventral ciliary apparatus of both daughters as a consequence of division: the same phenomenon occurs after conjugation. The process has been studied by Stein, Balbiani, Engelmann, Sterki,* and Prowazek,† and even yet we cannot claim to know all the details. I have attempted to follow the process in the case of *Stylonichia*, the favourite object of examination, but it is a work of exceptional difficulty. Previous to division, new frontal cirri are formed in the anterior half of the dividing creature, practically under the old: they are remarkably transparent and clear. Then follow in succession, according to Prowazek, the new adperistomal cirri, and finally the still insignificant anal cirri. Behind the marginal cirri the rudiments of their successors are formed at the same time; they have a very crowded appearance. "The new adoral membranellæ arise close behind the old, so that at a later stage the latter seems ruptured." This sentence settles the one point most in dispute among the older workers as to whether the adoral zone was renewed or not. Personally I have no doubt of the fact. Much the same can be also observed in what will be the posterior daughter. The new cirri appear first as cilia, and are very irregular and often violent in their motions, contrasting with

* "Beiträge zur Morphologie der Oxytrichinen," *Zeitschr. f. wiss. Zoologie*, Bd. 31, p. 29 ff.

† *Protozoenstudien: Arbeiten aus den Zoolog. Instituten des Universität. Wien*, Tom. xi., Heft. ii., 1895–1899.

the old ones, which are generally lifeless. Their area of origin is very much less than that which they eventually occupy. The new anal cirri, in the case of the posterior daughter, are formed above the old. The area occupied by the latter is marked off by a sort of furrow, and undergoes degeneration although there is no actual separation from the body of the creature—rather absorption.

In binary fission the nucleus is naturally the seat of the most complicated processes. Even when changes are first noticeable in the cytoplasm, it can be sufficiently well maintained on *a priori* grounds that there are previous changes in the nucleus, which, though invisible, are yet the inciting cause of those that are visible in the plasma. Macronucleus and micronucleus alike divide, the latter usually in advance of the former. It is not yet possible to state authoritatively whether, in the case of two micronuclei, the two halves of the same micronucleus go to one daughter, or whether it is a half of each of the micronuclei that go to form the daughter micronucleus in any one of the offspring.

With regard to the duration of the process, no definite rule can be laid down. I have noted the following periods for the forms named when under observation:—

Paramecium caudatum, $1\frac{1}{2}$–2 hours.
Stylonichia pustulata, 1–2 hours.
Lacrymaria olor, 1–$1\frac{1}{2}$ hours.
Spirostomum ambiguum, 1–2 hours.
Spirostomum teres, 1–2 hours.

As regards the rate of fission, it may be noted in the first place that it is by no means constant throughout the Ciliata, but varies with the species. Each species has its own normal rate of division depending upon its specific qualities. The following list of rates of division is based upon Maupas. I have ventured to modify it slightly, in accordance with my own results. The range of temperature on which I base the modifications is 16°–22° C.

Stylonichia pustulata, every 12–16 hours.
Euplotes patella, every 24 hours.
Onychodromus grandis, every 12 hours.
Oxytricha pellionella, every 8 hours.
Spirostomum ambiguum, every second day.
Spirostomum teres, every two or three days.

Paramecium aurelia, every 24 hours.
Paramecium caudatum, every 24 hours.
Colpidium colpoda, every 8 hours.

In the second place, we may note that the rate of fission depends intimately upon the food conditions to which the creatures are subjected. In conducting these experiments I have been in the habit of employing two distinct foods—either a hay infusion of a light straw colour, into which one put a piece of meat to hasten the production of bacteria, or else the forms produced by making a very dilute paste with ordinary flour and water. *Paramecium* and *Stylonichia* take kindly to either of these media, of which a drop was added daily to the slide on which they were isolated. The conditions were kept as constant as possible by the withdrawal of a couple of drops of the medium (some four or five drops in all) in which they had passed the night, which were replaced by one of food and another of distilled water. Pond water was also sometimes employed, but greater constancy was secured by the other method. I could not find that either of these two food media made any appreciable difference on the rate of division. But after a certain amount of manipulation one learned that there was a minimum of food that kept, *e.g.*, *Paramecia*, as they were; that there was also a definite amount, usually one drop, which caused one division in twenty-four hours; and that there was also a maximum which seemed to have an inhibitory effect upon the forms in question. In this case the body of, *e.g.*, *Stylonichia*, became positively black with unassimilated food matter, a condition of affairs that is reproduced in fig. 1. Change to a less rich medium soon resulted in a return to the normal state of affairs.

We may note in the third place that the rate of division bears a direct relation to temperature. To Maupas belongs the credit of having established this fact upon a comparatively sound basis. As far back as 1776 Spallanzani had observed that the multiplication of Ciliata was accelerated by increased temperature. But it is Maupas' chief count against the defective work of his predecessors that they had not properly attended—in some cases not at all—to the temperature and food conditions. The following rates of division under different temperatures are taken direct from Maupas.

Species.	5°–10° C.	10°–15° C.	15°–20° C.	20°–25° C.
Stylonichia myt.	48 h.	24 h.	12 h.	8 h.
Euplotes pat.	...	...	24 ,,	12 ,,
Onychodromus gr.	48 ,,	24 ,,	12 ,,	6 ,,
Oxytricha fallax	...	12 ,,	8 ,,	...
Colpidium colpoda	...	12 ,,	8 ,,	...
Glaucoma scintillans	8 ,,	6 ,,	5 ,,	...

Balbiani * has also suggested that the volume of water in which the Infusoria are kept has a direct influence upon their power of increase. Thus he says that *Paramecium aurelia* requires to be kept in a cubic centimetre of water in order to realise its full power of multiplication. In view of the interesting results that E. Warren has obtained with different bulks of water in the case of Daphnia (*Q. J. M. S.*, 1900), it would seem as if similar treatment of the Infusoria offered a field in which good results could be obtained.

It has long been known that the comparative regularity with which binary fission was carried out in favourable circumstances decreased as the period lengthened since the last conjugation. I have observed this phenomenon several times in the case of *Paramecium caudatum* and *Stylonichia pust.*, but have been unable to express the gradually decreasing energy in the terms of any formula. On the other hand, it has been maintained that immediately after coming out of conjugation these two forms show a marked increase in the rate of fission—the expression of a surplusage of energy; of this phenomenon, however, I have never seen any trace. So far as I am aware, conjugation results in no difference in the after-rate of ordinary multiplication, and this also would appear to be true of light and darkness.

Before proceeding to examine Maupas' theory of binary fission and the recent attack upon it, I should like to refer to his specification of two distinct species of *Paramecium aurelia* and *caudatum*.† He gives a definite account of these two forms, in which the latter is described as possessing an elongated body, as

* "Observ. et expér. s. les phénom. de la reprod. fissipare chez les infus. ciliées," *Compt. rend. Ac. Sc. Paris*, T. 50, p. 1191.

† No reference is made for the present to other species, *e.g.*, *putrinum*, *bursaria*.

being fusiform, obtuse in front and thinner behind. This species is also credited with one micronucleus, and the zygote nucleus was said to give rise to eight corpuscles. The other species had a broader body, was almost oval, and obtuse at both extremities. It further possessed two micronuclei, and the zygote nucleus gave rise to only four corpuscles. Since this account no special notice seems to have been taken of the two species, except to cast doubt upon their existence as two distinct species. Thus, in the *Zoologie Descriptive*, Fabre-Domergue states that neither he nor Balbiani have ever come across this *Paramecium* with the double micronucleus, and he makes the remark in such a way as to suggest that Maupas was drawing on his imagination in his description of it. Accordingly we find but one species—*caudatum*, with the single micronucleus—recognised generally in the text-books and other literature. There is no doubt, however, that these two distinct species do exist. Figs. 2 and 3 are photographs of the two species which give a very good idea of their relative sizes. Measurement of certain stained specimens which bring out the nuclear characteristics give *P. aurelia* a length of 80 μ and a breadth of 40 μ, while on the same scale *P. caudatum* has a length of 130 μ and a breadth of 50 μ. These figures, though hardly exact for the living form, bring out the peculiar feature of *aurelia* as compared with *caudatum*, viz., the high proportion that its breadth bears to its length. The magnification of the photographs is about 80. I may also mention here that I made frequent endeavours, through isolation of pairs, to get the two species to conjugate. The disproportion in size offered no *a priori* objection, as one often sees equal disproportion in the case of conjugating *Stylonichia*; and even in the case of *P. caudatum* the inequality is often marked. The representatives of *P. caudatum* were selected from a culture in which an epidemic of conjugation had set in; while the *aurelia* were taken from another culture which was far advanced in the number of its divisions. I never had the chance of contemporaneous epidemics amongst the two species, and accordingly always selected *P. caudatum* as the form that one certainly knew was ready for conjugation, inasmuch as it is the larger and probably more forceful species. Out of twenty-one attempts I had but two partial successes. Conjugation took place

on two slides: the period was normal. After separation each of the ex-conjugates divided once: on the third day they died off. In anticipation of something of this sort from analogy in higher forms, I intended to let the two pairs run their natural course, foregoing the desire to examine their nuclear condition. In view, therefore, of the incompleteness of the experiment, it is perhaps unwarrantable to draw any results regarding hybridisation and infertility, or even the "fixity of species," so far down in the animal scale.

As has been previously mentioned, numbers of observers have remarked that the comparative regularity with which binary fission proceeds under favourable circumstances decreases as the time increases since the last conjugation, and one has often wondered if it would not be possible to express this decrease by means of a mathematical curve or formula. In this connection, it is Maupas' chief distinction to have established that in the case of each species this power of binary fission comes to an end after a definite number of divisions; and that, were there no other method of restoring this potentiality to the individual, the species would come to an end. With the later stages of this gradual loss of fission-energy, he found distinct degeneration of the creature associated. In this degeneration he distinguished two well-marked stages. The first stage is not accentuated by any particular external change in the infusorian, unless, possibly, a slight reduction in size.* It continues to feed and multiply in the normal manner, but all the while it is giving rise to successors that are entering the second stage of degeneration. Moreover, when preserved in the ordinary method it is found to have undergone a certain atrophy of its nuclear apparatus. The macronucleus fragments (*Styl. pust.*), or may disappear altogether (*Styl. myt.*). The micronuclei are reduced to one, or even none (*Styl. pust.* and *Oxytricha sp.*). On the other hand, after such reduction they may later increase to numbers in excess of the norm (*Styl. myt.* and *Onychodromus gr.*). In the second stage of this senile degeneration the infusorian loses its power of multiplication. It no longer takes in food, and its body in consequence becomes quite clear. There is now a marked decrease in size,† and atrophy of external organs and appendages

* In the case of *Stylonichia pust.* this reduction varied from 25 to 50 μ.

† *Stylonichia pust.* now measures 70–90 μ in place of the normal 160 μ.

sets in. Finally, this degeneration further expresses itself in a sort of sexual hyperæsthesia, causing sterile conjugations that inevitably end in the death of the partners.

Now, Maupas determined that senile degeneration began in the case of *Styl. pust.* about the 170th generation or division, and that death ensued at the 316th. Similarly, cultures of *Onychodromus* became extinct after 330 generations, and so on. It is also an integral part of his theory that it is impossible to induce conjugation during the earlier bipartitions which cover a definite period of immaturity: in the case of *Styl. pust.* this extended to the 130th division. At the end of these earlier divisions—at the 131st in the case of *Stylonichia*—puberty is attained, and conjugation can be induced. This period of eugamy lasts over a definite number of divisions—until the 170th, as we know in the case of *Stylonichia*—when senile degeneration sets in, ending in death.

The first thing that strikes one when examining Maupas' tables of binary fission is their mechanical regularity. The following represents the first fortnight of the well-known *Stylonichia pustulata* table:—

Date.	Temperature.	Individuals.	Number of Bipartitions.	
			In 24 hours.	Total.
February 27	16°	1	...	...
,, 28	16	2	1	1
March 1	16	4	1	2
,, 2	16	32	3	5
,, 3	17	147	2	7
,, 4	18	483	2	9
,, 5	18	935	1	10
He isolated one of the 935.				
March 6	19	2	1	11
,, 7	19	8	2	13
,, 8	18	64	3	16
,, 9	17	230	2	18
He isolated one of the 230.				
March 10	17	4	2	20
,, 11	17	16	2	22
,, 12	16	126	3	25

I have taken the figures for the first fortnight, but greater regularity could have been shown if one had taken a fortnight at a later date. Fabre-Domergue confesses that he never succeeded in obtaining such regularity in any cultures that he undertook, and it seems to me that in so saying he intimates that he obtained a series of divisions that is much more natural than anything represented in Maupas' mathematical tables. When cultures of *Stylonichia* or *Paramecium* are kept in glass vessels where they may have some small bulk of water in which to live, they do not multiply at this rate, or with such regularity. It is not my intention, however, to impeach Maupas' tables as a whole, for with his results I find myself largely in agreement as against his latest adversary Joukowsky. Nevertheless, apart altogether from venturing to inquire how such exactness was acquired as is expressed in 935 Stylonichia, I would maintain that the results which Maupas first established are reached by a process of division that is far from regular, and depends to a great extent upon the individuality of the infusorian. Even under the happiest possible conditions (so far as one can judge), artificial or natural, binary fission does not proceed with that constant regularity that the French savant would ascribe to it. The following table, representing a few weeks of a short series, expresses, I believe, a more natural rate of progress than one would gather to be the case from Maupas' table. The form experimented with was *Paramecium caudatum*, and in every case the series was commenced with two exconjugates. I have reckoned that case as one bipartition in 24 hours, where half or more of the creatures on the slide divided.

[TABLE.

Date.	Temperature.	Slide.	Individuals.	Number of Bipartitions. In 24 hours.	Number of Bipartitions. Total.
June 13	16°	a	2	0	0
		c	2	0	0
		f	2	0	0
		g	Pair still in conjugation.		
		h	,,	,,	
		q	...	...	...
,, 14	16	a	4	1	1
		c	3	1	1
		f	4	1	1
		g	2	0	0
		h	2	0	0
		q	Pair in conjugation.		
,, 15	17	a	4	0	1
		c	4	0	1
		f	4	0	1
		g	2	0	0
		h	4	1	1
		q	2	0	0

An increase in size was apparent in the case of *a* and *g*.

Date.	Temperature.	Slide.	Individuals.	In 24 hours.	Total.
June 16	20	a	5	0	1
		c	4	0	1
		f	7	1	2
		g	2	0	0
		h	4	0	1
		q	2	0	0
,, 17	21	a	8	1	2
		c	6	1	2
		f	11	1	3
		g	2	0	0
		h	8	1	2
		q	3	1	1
,, 18	23	a	10	0	2
		c	8	0	2
		f	22	1	4
		g	3	1	1
		h	11	0	2
		q	3	0	1
,, 19	21	a	13	0	2
		c	10	0	2
		f	31	0	4
		g	5	1	2
		h	21	1	3
		q	5	1	2
,, 20	20	a	16	0	2
		c	15	1	3
		f	32	0	4
		g	8	1	3
		h	27	0	3
		q	7	0	2

Date.	Temperature.	Slide.	Individuals.	Number of Bipartitions. In 24 hours.	Number of Bipartitions. Total.
June 21	23°	a	20	0	2
		c	27	1	4
		f	52	1	5
		g	8	0	3
		h	42	1	4
		q	7	0	2

With the exception of *g* and *q* five *Paramecia* were removed from each slide.

Date.	Temperature.	Slide.	Individuals.	In 24 hours.	Total.
June 22	20	a	30	1	3
		c	36	1	5
		f	70	1	6
		g	9	0	3
		h	51	0	4
		q	14	1	3
,, 23	19	a	45	1	4
		c	36	0	5
		f	72	0	6
		g	18	1	4
		h	70	0	4
		q	14	0	3
,, 24	20	a	45	0	4
		c	45	0	5
		f	73	0	6
		g	18	0	4
		h	70	0	4
		q	18	0	3
,, 25	23	a	63	0	4
		c	51	0	5
		f	76	0	6
		g	36	1	5
		h	70	0	4
		q	20	0	3
,, 26	24	a	81	0	4
		c	55	0	5
		f	77	0	6
		g	44	0	5
		h	86	0	4
		q	25	0	3
,, 27	24	a	81	0	4
		c	56	0	5
		f	88	0	6
		g	70	1	6
		h	104	0	4
		q	37	1	4

The numbers on these slides were reduced to 13, 7, 14, 14, 10, and 7 respectively.

Date.	Temperature.	Slide.	Individuals.	Number of Bipartitions. In 24 hours.	Total.
June 28	23°	a	26	1	5
		c	7	0	5
		f	20	0	6
		g	27	1	7
		h	18	1	5
		q	11	1	5
,, 29	23	a	45	1	6
		c	9	0	5
		f	27	0	6
		g	42	1	8
		h	25	0	5
		q	16	1	6
,, 30	23	a	53	0	6
		c	14	1	6
		f	34	0	6
		g	46	0	8
		h	38	1	6
		q	23	0	5
July 1	22	a	90	1	7
		c	28	1	7
		f	60	1	7
		g	77	1	9
		h	54	1	7
		q	23	0	5
The numbers on the first five slides were reduced to 43, 9, 13, 14, and 7 respectively.					
July 2	20	a	64	1	8
		c	18	1	8
		f	16	0	7
		g	28	1	11
		h	12	1	8
		q	27	0	5
,, 3	18	a	99	1	9
		c	23	0	8
		f	20	0	7
		g	51	1	12
		h	15	0	8
		q	34	0	5
,, 4	18	a	99	0	9
		c	25	0	8
		f	30	1	8
		g	51	0	12
		h	15	0	8
		q	34	0	5

Here, then, in a period of three weeks, with a temperature ranging from 16° to 24° C., there is, over six slides, an average of eight divisions. This series was by no means the first that I inaugurated, and the slides were numbered from *a* to *r*. The

majority were treated daily in a constant manner, given the same definite amount of food (calculated from previous experiments), and provided with a certain amount of fresh water at definite intervals. With the others I experimented in the amount of food given, in the period of time that they were left without any change of water, in the amount of water on the slide, and so on. In no single instance did I obtain such clockwork regularity as Maupas' tables show. The slides, whose history is given, were amongst those that were treated with regularity so far as I was able, and consequently they were all treated alike. Hence a day like June 25, when but one of the slides shows a complete Maupasian division, appears to me to represent the more natural state of affairs, and for no reason more than this. Maupas' table of *Stylonichia pustulata* admittedly deals with a form that multiplies more quickly than *Paramecium caudatum*—possibly about twice as an average over all temperatures. But at the close of the first three weeks his Stylonichia has divided no fewer than thirty-nine times with a temperature that ranged from 15° to 19° C. Now, no one has laid more stress upon the influence of temperature in raising the rate of division than Maupas, and yet I do not find from the table that his high rates of division bear any relation to the temperature. Quite the contrary is the case, for out of the four occasions within the first three weeks on which the Stylonichia divided three times in 24 hours, on two of them the temperature was actually a degree lower than the previous day, when it divided a less number of times. Accordingly, although I believe that ultimately continued binary fission involves a certain degeneration, and that Maupas' theory of the matter is largely correct, still it is altogether false to imagine that under natural conditions a Stylonichia will rush through 316 divisions in 4½ months: that is to say, the validity of Maupas' method is open to question, and where this is so the results are always in jeopardy more or less.

I may refer here to two peculiar cases of division that came under my notice. On 13th May 1900 two paramecian exconjugates were isolated on a slide and subjected to ordinary culture treatment. On the 14th they were as before; on the 15th there were three; they remained at this number on the 16th and

presented no abnormality.* On the 17th there were four *Paramecia* on the slide, but one of them had developed a cleft tail. The cleft was in a plane perpendicular to the dorso-ventral axis. From the first it extended to a depth of about 25 μ and did not grow deeper. Otherwise the creature appeared to be perfectly normal: the two contractile vacuoles functioned and the internal circulation swept round, clear of the divided tail. On the 18th it remained as it was. On the 19th it had divided, and the anterior half, though distinctly undersized and resembling rather the species *aurelia* in configuration, was yet normal in every other respect and continued afterwards to divide by itself. On the 20th the original form again divided, but not on the 21st. However, on the 22nd it resumed operations, and while the anterior one still retained its aurelian characters, I noticed that the internal circulation of the posterior half no longer swept clear of the tail, but had partially entered into one (the dorsal) lobe, which now contained excretory granules and one or two small food vacuoles. At the same time this lobe had slightly increased in size, while the other had correspondingly decreased. On the 23rd it divided again, but not on the 24th. The anterior parts still retained the same characteristics as formerly, and gave rise themselves to otherwise normal *Paramecia.* In the posterior cleft-tail *Paramecium* the dorsal lobe continued to grow, while the other was more and more absorbed. The mouth also was driven unusually far back. On the 25th and 26th it again divided, but not on the 27th. The ventral lobe had now been completely absorbed, while the other had increased in size till it now measured some 60 μ; there was, however, no proportionate increase in breadth, and there seemed to be a tendency for it to get blocked. At any rate the number of excretory granules increased, and the circulation slowed down. On the 28th and 29th it again divided, but the dorsal tail seemed thoroughly congested, and by the 30th it was dead. None of the daughters reproduced the peculiarity in themselves or in their descendants.

The other case is still more peculiar. About the same time as the preceding exconjugates were isolated—one of several

* No temperatures are given, as in this case they probably had no influence on the sequel.

series—on another slide two had been set aside, which on the following day were found to be still, in a sense, the same number. Yet evidently one of the creatures had begun to divide, but stopped in the process, so that while on the slide there was one normal form, the other was a monster, composed of two full-grown *Paramecia* with organic union between the anterior part of what should have been the posterior daughter and the posterior part of the anterior one. There was no constriction between the two, or other hint of their origin. The two bodies formed one continuous whole with one circulation, and was so flexible throughout that the two extremities could touch. On the 17th of May it appeared, and on the 18th was in no way changed. On the 19th the slide on which it was isolated contained three forms: the monster had given off a daughter from either end. These daughters were ordinary *P. caudatum* of good average size; they continued to divide by themselves, and in every way appeared to be normal. On the 20th the monster remained as it was, but again on the 21st it repeated the operation of giving off a daughter from either end. On the 22nd it had not multiplied, but on the 23rd for the third and last time it had given off a daughter from either end. These last, however, were markedly smaller in size, and otherwise like the "aurelian" daughters that had been given off by the cleft-tail individual. Previous to this the posterior creature had gradually been becoming inclined at an angle to the anterior one. Up to this point the combined activity of the monster had been as great as that of any normal *Paramecium*. The anterior half, perhaps naturally, was the more active, and, in a sense, the guiding part. Its cilia were feverishly active: they were also longer and better developed, especially in the anterior regions, than those of the posterior creature. This greater anterior activity may also have found expression in a process that began to come off from it a little above the angle made with the posterior form. Further, the two contractile vacuoles of the anterior creature were close together and contracted simultaneously. From the 23rd, *i.e.*, about a week after its appearance, growth ceased to show itself, as we have seen, in the regular separation from either end of two daughters on every second day, and began rather to express itself in the growth of the aforesaid process and in remarkable

lateral expansion of the anterior half. This and an earlier stage are shown in fig. 4. In this peculiar condition it remained with slight modifications about another week, but was dead by the 28th.

Up till quite recently Maupas' classical work has been permitted to go comparatively unchallenged. In the *Verhandlungen des Naturhistorisch-Medizinischer Vereins zu Heidelberg*, however, D. Joukowsky publishes certain "Beiträge zur Frage nach den Bedingungen der Vermehrung und des Eintritts der Konjugation bei den Ciliaten," which go contrary somewhat to the received views.

Joukowsky's observations were made upon *Pleurotricha lanceolata*—a form allied to Stylonichia—*Paramecium caudatum*, and *Paramecium putrinum*. He says that he got irregular divisions at first: only after a month did the forms divide regularly. After the numbers on a slide had reached one hundred, division was slower. I have already referred to this question of regular and irregular division. My own experiments were more than once carried on considerably over two months, and I did not find any greater regularity after the first four weeks than I did during that time. Nor is it easy to see why this should be so. To imagine that these infusorians will settle down after a month into regular methods of division simply means failure to appreciate the conditions of the experiment. Joukowsky indeed says that the abnormalities in the division rate were due to the abnormal relations in which the creatures live. Bacteria generate and hinder ordinary division, and one may well suppose that the secretions and excretions of the creatures themselves may be ultimately dangerous in such a circumscribed area. But then this investigator deliberately states that after a month the divisions became regular; and yet we are not led to believe that he had found any means of overcoming the difficulties in which he sees the cause of the earlier irregular divisions. Obviously, therefore, they cannot have played the part that he imagines. I may also mention here that Maupas, while making these largely statistical experiments in binary fission, employed cover-glasses on his slides in the damp chamber. This appears to me to have been the introduction of an altogether unnecessary artificial condition. So far as regards the observation of the mere rate and other simple

aspects of binary fission, I never employed cover-glasses: any infusorian requiring high-power examination was easily isolated. Maupas also states that in his damp chamber there was very slight evaporation, and that "when it was necessary" he made compensation for the loss with rain water. If he added food daily, it is difficult to see how it was never necessary in addition to make up for evaporation. If the latter had to be done at all, it were surely better to change the water in greater or in less quantity with regularity, and so give less occasion to bacteria to generate. I cannot say that I found the latter method unsuccessful when I tried it.

Joukowsky kept *Pleurotricha lanceolata* for a period of eight months, in the course of which 458 divisions occurred, and during that time he got neither conjugation—not even when he starved the creatures and set them in pure water—nor evidence of degeneration. In a certain degree there is correspondence here with Maupas' experiments on *Stylonichia mytilus*, where senile degeneration (which, however, Joukowsky did not find) did not seem to stimulate this species to conjugation as it did in the case of *Stylonichia pustulata*. Joukowsky, nevertheless, observed a certain shrinkage in size, which he found depended on the quantity and quality of food. The following is his temperature table :—

	30° C.	23° C.	15° C.
13 xii. 1894 : 6 p.m.	1 individual.	1 individual.	1 individual.
14 xii. 1894 : 6 p.m.	16 individuals.	8 individuals.	2 individuals.

The question of degeneration is probably the most important that he raises. As we have already seen, Maupas distinctly states that at the end of the period of eugamy, which covers a definite number of divisions of the creature, senile degeneration sets in, which ends in death if conjugation does not intervene : we have also seen the method in which this degeneration expresses itself. On this subject he had already been challenged by Bütschli, who maintained that the fission capacity of the Ciliata was specially great and much in evidence after conjugation, but that thereafter it gradually ebbed away. If by this Bütschli meant that immediately after conjugation the rate of fission is above the normal, I can only say that I have never observed anything of this nature in the several forms that have come under my observation. But if, as

seems more evident (*Protozoa*, Bd. I. Heft III. p. 1592), he is simply entering a protest against Maupas' action in limiting the process of degeneration to one special late period in the infusorian's life—thus in the case of *Stylonichia pustulata* it is not reached until from the 170th to the 200th generation—he is surely to be commended. If there is degeneration at all, it is most improbable, on all other analogy, that it should set in at a certain fairly definite point—so late as the last third of the creature's life. If there is degeneration, it has commenced invisibly long before those outward manifestations in the loss of frontal cirri and other appendages; it is ever so with decay. And in referring to Maupas' *Stylonichia* series, with its increasing temperature from the middle of the period onwards, and bearing in mind the effect that temperature has on the rate of fission, Bütschli is only asking a common-sense question when he demands how, under these conditions, it could have been possible to recognise a gradual ebb in the fission-energy, such as we may suppose to constitute the initial stages of degeneration. Joukowsky, then, found no degeneration in his eight-months cultured *Pleurotricha*. He never saw the disappearance of the frontal membranellæ: he found no abnormal relations in the condition of the nuclei, unless in two cases, when a certain change in the relative positions of the two parts of the macronucleus was noted. He examines Maupas' *Stylonichia* table, and finds that the creature multiplied much more quickly in the later weeks,* and to this he in large part attributes the degeneration. "It is very possible that the cause of the degeneration which Maupas observed is not the mere number of generations alone, but the number of generations in association with the rapidity of multiplication." For my own part I have looked for evidence of degeneration throughout 3–4 month slide cultures† of both the *Paramecia* and *Stylonichia pustulata*, as also in the case of other odd forms that I happened to find in quantity *previous to* an epidemic of conjugation, but have not recognised it in such specific manner as nuclear degeneration or loss of external

* Some four or five times every 24 hours in place of the normal twice or thrice.

† In some cases these covered the period of eugamy as calculated by generations.

appendages. Still, none the less am I convinced of a gradual ebbing of vital energy as the series proceeds, which expresses itself in slowed motion, in a tendency to inactivity and general listlessness (if the word be admissible in this connection), as also in a certain diminution in size that was not remedied by any amount of food.

Joukowsky also made observations on a culture of *Paramecium caudatum*. In a temperature of 19°–23° C. he got them to divide one or two times. By the seventh month he noticed that they divided badly. Some of the individuals seemed dead, but on examination they were found to be still alive. The cilia on the upper surface had almost completely disappeared; indeed it was only at either end and in the region round about the mouth that he found ciliation at all. He made out, however, no hint of nuclear degeneration.

Maupas laid great stress on the period of immaturity in the infusorian's life—that definite number of divisions previous to puberty that had to be gone through before it was in a fit state to conjugate. We saw, *e.g.*, that this period was reached by *Stylonichia pustulata* at the 130th division. Joukowsky, experimenting with *Paramecium putrinum*, found that this period of puberty was attained after some seven or eight divisions, that is to say, it is practically always present. In this particular species he succeeded in getting exconjugates to conjugate within that small number of divisions, and maintains in consequence that Maupas' rule does not have universal validity. Now it is well known that by means of starvation not only can Ciliata be prevented from multiplying by binary fission, but after they have reached the period of puberty they can be hurried into conjugation by a similar method.* I therefore made deliberate attempts, by means of starvation and other unfavourable means, in another series similar to that of which details have already been given, to induce conjugation within the period of puberty, but never succeeded. The forms experimented with were the two Paramecia (*aurelia* and *caudatum*), *Stylonichia*

* With regard to the former point, we may note that those Ciliata that have been hindered in this way from reproducing themselves by binary fission require some little time to recover the power to do so when food is again supplied to them.

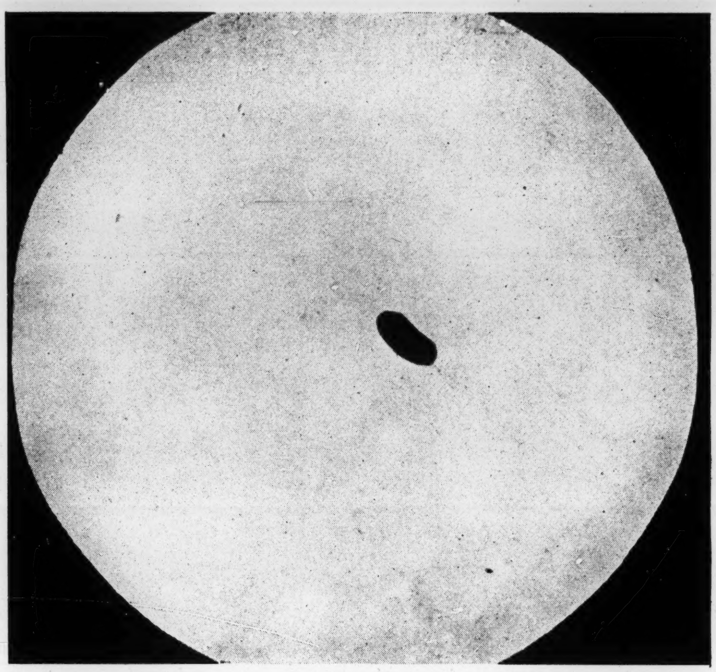


Fig. 1.—Overfed Stylonichia pustulata (see text). It is black with excretory granules.

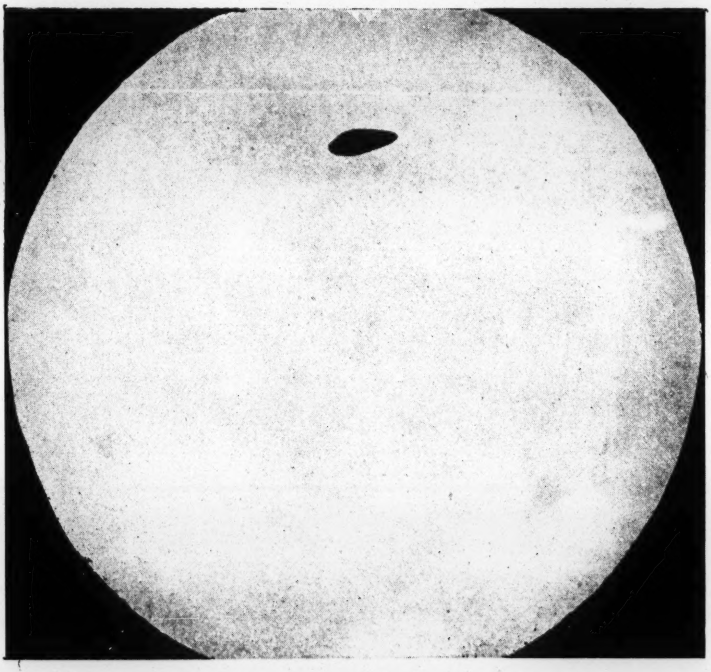


Fig. 2.—Paramecium aurelia (× 80). *Cf.* with fig. 3.

J. Y. Simpson.—Plate I.

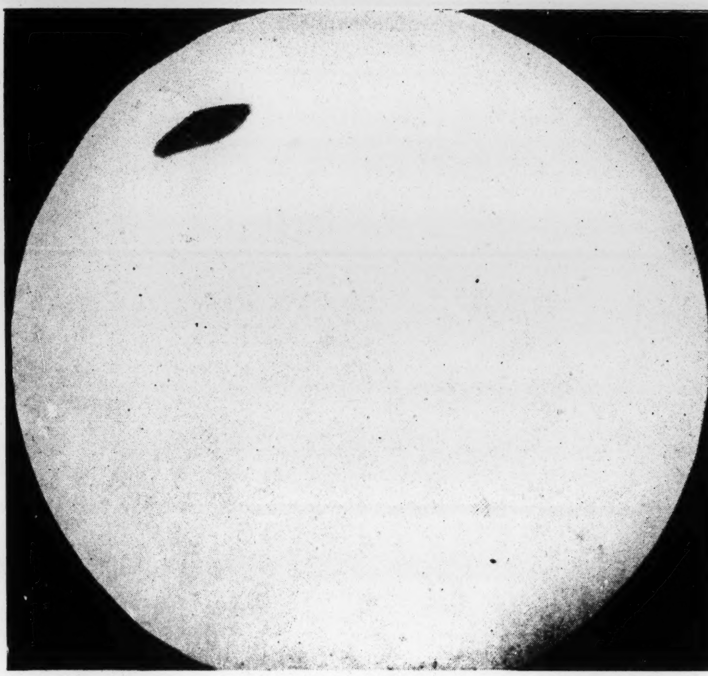


Fig. 3.—Paramecium caudatum, dorsal aspect (× 80). *Cf.* with fig. 2.

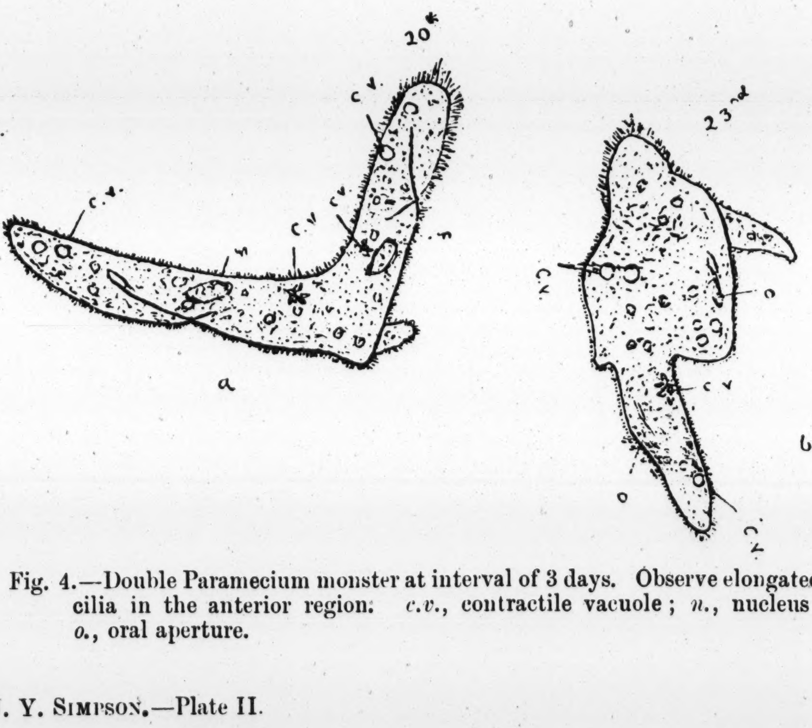


Fig. 4.—Double Paramecium monster at interval of 3 days. Observe elongated cilia in the anterior region. *c.v.*, contractile vacuole ; *n.*, nucleus ; *o.*, oral aperture.

J. Y. Simpson.—Plate II.

pustulata, and *Oxytricha pellionella*. It is of course true that ever so many negative results do not contradict one positive result, but I must confess that I am extremely doubtful concerning this phase of Joukowsky's work, and am entirely in agreement with Maupas' view.*

The French biologist also maintained that conjugations between near relatives were sooner or later sterile. As I have already shown, this is possibly true in the case of conjugation between members of the two *species* of Paramecium. But in the case of *Paramecium putrinum*, Joukowsky observed effective conjugation between the descendants of one individual; at the same time he admits that this probably has its limits. I also am inclined to believe that this peculiar process has its limits—but in the Maupasian sense; for although I have observed conjugation in the case of *P. caudatum* between the descendants of an exconjugate, in the four or five instances in which I kept them they all died off within four to eight divisions.

* Joukowsky fails to observe that in *Leucophrys patula* and *Paramecium putrinum* Maupas recognised possible exceptions to his puberty theory (*Le Raj. karyog. chez les Ciliés*, p. 410), while he also admits that the period of immaturity may be greatly shortened under certain unknown conditions. Of these conditions I have been able to find out nothing.

On the Thermo-electric Properties of Solid Mercury.

By Dr W. Peddie and the late Alex. B. Shand, Esq.

(Read February 18, 1901.)

(Abstract.)

This paper contained an account of a redetermination of the thermo-electric position of solid mercury, by a method described in a note read last session. The only difference was that, by the use of three galvanometers, *simultaneous* readings of the deflection due to the Hg-Fe circuit, and of the deflections due to thermo-electric circuits giving the temperatures of the two Hg-Fe junctions, were taken.

The results confirmed those previously obtained.

It may be said that the line of solid mercury on the thermo-electric diagram is practically parallel to the iron line at ordinary temperatures, and that, if produced, it meets the line of copper at or near, its intersection with the ordinate of 0° C.

Note on a Proposition given by Jacobi in his "De determinantibus functionalibus." By Thomas Muir, LL.D.

(Read July 1, 1901.)

(1) The proposition in question is stated as follows * :—
"*Ponamus (enim) inter quantitates,* x, x_1, . . ., x_n *datas esse totidem aequationes*

$$f=a, \quad f_1=a_1, \quad \dots, \quad f_n=a_n,$$

in quibus a, a_1, . . . *sint Constantes: dico Determinans*

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdot \dots \cdot \frac{\partial f_n}{\partial x_n}$$

non mutare valorem si functiones f, f_1, . . ., f_n *varias subeant mutationes quales per aequationes propositas subire possunt, ita tamen ut functioni alicui* f_i *transmutandae non ipsa adhibeatur aequatio* $f_i = a_i$."

If we were dependent on this alone for Jacobi's meaning there might be some difficulty in regard to the interpretation. Fortunately, however, at the conclusion of his demonstration he restates the proposition in another form, viz. "Si per aequationes

$$f=a, \quad f_1=a_1, \quad \dots, \quad f_{i-1}=a_{i-1}, \quad f_{i+1}=a_{i+1}, \quad \dots, \quad f_n=a_n$$

fiat

$$f_i=\phi_i\,;$$

per aequationes

$$f=a, \quad f_1=a_1, \quad \dots, \quad f_n=a_n$$

fore

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdot \dots \cdot \frac{\partial f_n}{\partial x_n} = \sum \pm \frac{\partial \phi}{\partial x} \cdot \frac{\partial \phi_1}{\partial x_1} \cdot \dots \cdot \frac{\partial \phi_n}{\partial x_n}."$$

(2) The expression "Ponamus inter quantitates x, x_1, . . , x_n datas esse totidem aequationes $f=a$, $f_1=a_1$, . . ., $f_n=a_n$ in quibus a, a_1, ... sint Constantes" is particularly unfortunate, for it is certainly not intended that $n+1$ equations are given, by the solution of which the independent variables x, x_1, . . ., x_n may be shown to be constants! In fact, a, a_1, . . . are simply alternative symbols for f, f_1, . . ., two symbols being deemed desirable for each function because the function requires to be

* *Crelle's Journ.*, xxii. p. 345.

viewed in one set of differentiations as being dependent on $x, x_1, \ldots$, and in another set as being constant with respect to $x, x_1, \ldots$ This amounts to saying that it might be preferable to write $f \equiv a$ instead of $f = a$.

(3) In the next place, by "the change of f_i into ϕ_i by means of the equations $f = a$, $f_1 = a_1$, $\ldots$, $f_{i-1} = a_{i-1}$, $f_{i+1} = a_{i+1}$, $\ldots$; $f_n = a_n$" we are to understand the performance of substitutions whereby $a, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ make their appearance in the expression of f_i, this new expression, called ϕ_i, being therefore such that by resubstituting in it for $a, a_1, \ldots$ we are led back to f_i. After the second enunciation above quoted Jacobi himself says "Nimirum restituendo in omnibus $\frac{\partial \phi_1}{\partial x_\kappa}$ pro Constantibus $a, a_1, a_2, \ldots, a_n$ functiones $f, f_1, f_2, \ldots, f_n$ Determinans functionale alterum in alterum identice redit."

(4) The proposition may therefore be conveniently enunciated as follows without making use of the a's:—*If* f, $f_1 \ldots, f_n$ *be functions of* x, $x_1, \ldots, x_n$, *and by legitimate operations the functions* f, $f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n$ *be introduced into the expression for* f_i *which thereby takes the form of* ϕ_i, *then*

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots \cdot \frac{\partial f_n}{\partial x_n} = \sum \pm \frac{\partial \phi}{\partial x} \cdot \frac{\partial \phi_1}{\partial x_1} \cdots \cdot \frac{\partial \phi_n}{\partial x_n},$$

it being understood that in the differentiations of $\phi, \phi_1, \ldots, \phi_n$ *we are to view* f, $f_1, \ldots, f_n$ *as constants.*

(5) Jacobi's proof for the case where only one of the functions is changed, viz. f into ϕ, is irrefragable. He says, in effect, that as ϕ is a function of $x, x_1, \ldots, x_n, f_1, \ldots, f_n$, and f is what ϕ becomes when for $f_1, \ldots, f_n$ we substitute their expressions in terms of $x, x_1, \ldots, x_n$, it follows that

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial f_1} \cdot \frac{\partial f_1}{\partial x} + \frac{\partial \phi}{\partial f_2} \cdot \frac{\partial f_2}{\partial x} + \cdots + \frac{\partial \phi}{\partial f_n} \cdot \frac{\partial f_n}{\partial x}, \\
\frac{\partial f}{\partial x_1} &= \frac{\partial \phi}{\partial x_1} + \frac{\partial \phi}{\partial f_1} \cdot \frac{\partial f_1}{\partial x_1} + \frac{\partial \phi}{\partial f_2} \cdot \frac{\partial f_2}{\partial x_1} + \cdots + \frac{\partial \phi}{\partial f_n} \cdot \frac{\partial f_n}{\partial x_1}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial f}{\partial x_n} &= \frac{\partial \phi}{\partial x_n} + \frac{\partial \phi}{\partial f_1} \cdot \frac{\partial f_1}{\partial x_n} + \frac{\partial \phi}{\partial f_2} \cdot \frac{\partial f_2}{\partial x_n} + \cdots + \frac{\partial \phi}{\partial f_n} \cdot \frac{\partial f_n}{\partial x_n}.
\end{aligned}$$

Using these equivalents for $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ we transform the Jacobian

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdot \cdots \cdot \frac{\partial f_n}{\partial x_n}$$

into

$$\begin{vmatrix} \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial f_1} \cdot \frac{\partial f_1}{\partial x} + \ldots, & \frac{\partial f_1}{\partial x}, & \frac{\partial f_2}{\partial x}, & \ldots, & \frac{\partial f_n}{\partial x}, \\ \frac{\partial \phi}{\partial x_1} + \frac{\partial \phi}{\partial f_1} \cdot \frac{\partial f_1}{\partial x_1} + \ldots, & \frac{\partial f_1}{\partial x_1}, & \frac{\partial f_2}{\partial x_1}, & \ldots, & \frac{\partial f_n}{\partial x_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \phi}{\partial x_n} + \frac{\partial \phi}{\partial f_1} \cdot \frac{\partial f_1}{\partial x_n} + \ldots, & \frac{\partial f_1}{\partial x_n}, & \frac{\partial f_2}{\partial x_n}, & \ldots, & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

and this on having its first column diminished by multiples of the other columns becomes

$$\sum \pm \frac{\partial \phi}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n}$$

as was to be proved.

(6) Jacobi then proceeds with the case where *two* of the functions are altered, his exact words being—

"Si per aequationes

$$\phi = a, \quad f_2 = a_2, \quad f_3 = a_3, \quad \ldots, \quad f_n = a_n$$

fit

$$f_1 = \phi_1,$$

eodem modo probas fieri

$$\sum \pm \frac{\partial \phi}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdot \cdots \cdot \frac{\partial f_n}{\partial x_n} = \sum \pm \frac{\partial \phi}{\partial x} \cdot \frac{\partial \phi_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n}$$

unde etiam

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdot \cdots \cdot \frac{\partial f_n}{\partial x_n} = \sum \pm \frac{\partial \phi}{\partial x} \cdot \frac{\partial \phi_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n}."$$

This practically concludes his reasoning, for he merely adds "Sic pergendo sequitur generaliter . . . .", and gives the second of the two enunciations above quoted.

(7) Now what he here really proves is—*If* f, f_1, . . . , f_n *be functions of* x, x_1, . . . , x_n *and by legitimate operations the functions* f_1, . . . , f_n *be introduced into the expression for* f *which*

thereby takes the form ϕ, *and* ϕ, f_2, f_3, . . . , f_n *be introduced into the expression for* f_1 *which thereby takes the form* ϕ_1, *then*

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdot \cdot \cdot \cdot \frac{\partial f_n}{\partial x_n} = \sum \pm \frac{\partial \phi}{\partial x} \cdot \frac{\partial \phi_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdot \cdot \cdot \frac{\partial f_n}{\partial x_n}.$$

In other words, instead of stipulating that ϕ, f_2, f_3, . . ., f_n be introduced into f_1 he merely stipulates that f, f_2, f_3, . . ., f_n be introduced. His proof is thus defective.

(8) The nature of the oversight is possibly made clearer by observing what

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdot \cdot \cdot \frac{\partial f_n}{\partial x_n}$$

becomes, when, in addition to substituting for

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n},$$

as was done in the first case, we also substitute for

$$\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial x_1}, \ldots, \frac{\partial f_1}{\partial x_n}.$$

Even the first column of the altered Jacobian cannot now be simplified to the same extent as before, because part of the simplification consisted in subtracting a multiple of the second column in its unaltered form. In fact the result instead of being

$$\sum \pm \frac{\partial \phi}{\partial x} \cdot \frac{\partial \phi_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdot \cdot \cdot \frac{\partial f_n}{\partial x_n}$$

is

$$\begin{vmatrix} \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial f_1} \cdot \frac{\partial f_1}{\partial x} & \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_1}{\partial f} \cdot \frac{\partial f}{\partial x} & \frac{\partial f_2}{\partial x} & \frac{\partial f_3}{\partial x} & \cdots & \frac{\partial f_n}{\partial x} \\ \frac{\partial \phi}{\partial x_1} + \frac{\partial \phi}{\partial f_1} \cdot \frac{\partial f_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_1} + \frac{\partial \phi_1}{\partial f} \cdot \frac{\partial f}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_3}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \phi}{\partial x_n} + \frac{\partial \phi}{\partial f_1} \cdot \frac{\partial f_1}{\partial x_n} & \frac{\partial \phi_1}{\partial x_n} + \frac{\partial \phi_1}{\partial f} \cdot \frac{\partial f}{\partial x_n} & \frac{\partial f_2}{\partial x_n} & \frac{\partial f_3}{\partial x_n} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}.$$

(9) As an example let us take the case where

$$u_1 = x(y+z), \quad u_2 = y(z+x), \quad u_3 = z(x+y),$$

and where therefore

$$J(u_1, u_2, u_3) = \begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix} = 4xyz.$$

Altering u_1 by introducing into it u_2 and u_3 we have

$$u_1 = u_2 + u_3 - 2yz, \quad u_2 = y(z+x), \quad u_3 = z(x+y)$$

the Jacobian of which is

$$\begin{vmatrix} . & -2z & -2y \\ y & z+x & y \\ z & z & x+y \end{vmatrix}.$$

That this is the same as the previous Jacobian is readily seen by increasing its first row by the sum of the second and third rows.

If now, however, we alter u_2 and u_3 in the same way as u_1, we have

$$u_1 = u_2 + u_3 - 2yz, \quad u_2 = u_3 + u_1 - 2zx, \quad u_3 = u_1 + u_2 - 2xy,$$

and the Jacobian becomes

$$\begin{vmatrix} . & -2z & -2y \\ -2z & . & -2x \\ -2y & -2x & . \end{vmatrix}$$

which is not $4xyz$ but $-16xyz$. In the sense here given to it, therefore, Jacobi's proposition does not hold when more than one of the functions is changed.

Meetings of the Royal Society—Session 1899–1900.

The 117th Session.

General Statutory Meeting. Election of Office-Bearers. p. 1.

FIRST ORDINARY MEETING.

Monday, 4*th December* 1899.

The Right Hon. Lord Kelvin, President, in the Chair.

The Chairman gave an opening Statement, pp. 1–11.

The following Communications were read:—

1. On the Rectal Gland of the Elasmobranchs. By Dr J. Crawford. Communicated by Dr Noël Paton. pp. 55–61.
2. Obituary Notice of Charles Hayes Higgins, M.D. By Dr Sydney Marsden.
3. Further Investigations of the Life-History of the Salmon in Fresh Water. By Dr Noël Paton and M. I. Newbigin, D.Sc. pp. 44–54.
4. On the Eliminant of a Set of General Ternary Quadrics. Part II. By Thomas Muir, LL.D. *Trans.*, vol. 40, pp. 23–38.

Dr John Penny, Dr John Henderson, Professor Graham Lusk, Mr Alfred C. Wilson, Dr John W. H. Eyre, and Mr James Bisset were balloted for, and declared duly elected Fellows of the Society.

SECOND ORDINARY MEETING.

Monday, 18*th December* 1899.

The Rev. Prof. Duns, D.D., Vice-President, in the Chair.

The following Communications were read:—

1. The Presence of Enzymes in Normal and Pathological Tissues. By John Soutar M'Kendrick, M.D. Communicated by Professor M'Kendrick. pp. 68–89.
2. On the Convection of Heat by Air-Currents. By Professor A. Crichton Mitchell, D.Sc. *Trans.*, vol. 40, pp. 39–47.

3. A new Form of Myograph, and its Uses. By S. C. MAHALANOBIS, B.Sc., F.R.M.S., F.R.S.E., Assistant Lecturer on Physiology, University College, Cardiff. pp. 62–67.

4. On Swan's Prism Photometer, commonly called Lummer and Brodhun's Photometer. By Dr C. G. KNOTT. pp. 12–14.

5. On the Claim recently made for Gauss to the *Invention* (not the *Discovery*) of Quaternions. By Professor TAIT. pp. 17–23.

6. Professor Klein's View of the Nature of a Quaternion. By Dr C. G. KNOTT. pp. 24–34.

THIRD ORDINARY MEETING.

Monday, 8th January 1900.

Sir William Turner, LL.D., D.C.L., Vice-President, in the Chair.

Mr ALFRED C. WILSON was admitted a Fellow of the Society.

The following Communications were read:—

1. Two Historical Fallacies:—Heather Beer, and Uisge Beithe. By Dr W. CRAIG MACLAGAN. *Trans.*, vol. 40, pp. 15–22.

2. On the Thermo-electric Properties of Solid and Liquid Mercury. By Dr W. PEDDIE and Mr A. B. SHAND. p. 15.

3. On the Azores Bank, and some recent Deep-sea Soundings in the North Atlantic. By A. E. PEAKE, Esq., M.Inst.C.E., and Sir JOHN MURRAY, K.C.B.

4. The Examination of Sea-Water by an Optical Method. By JOHN J. MANLEY, Esq. Communicated by Sir JOHN MURRAY, K.C.B. pp. 35–43.

FOURTH ORDINARY MEETING.

Monday, 22nd January 1900.

The Rev. Prof. Duns, D.D., Vice-President, in the Chair.

The following Communications were read:—

1. The Torsional Constants of Iron and Steel. By Dr W. PEDDIE. p. 16.

2. Simple Proof of Gibbs' Phase-Rule. By Professor KUENEN, University College, Dundee. pp. 317–318.

3. Change of the Coefficient of Absorption of a Gas in a Liquid with Temperature. By the Same. pp. 312–316.

4. On the "Cosmosphere," an instrument for exhibiting Astronomical and Navigational Problems in a concrete form:—and on a Slide-Rule for solving, by inspection, Astronomical and Navigational Problems. By WALTER B. BLAIKIE, Esq.

FIFTH ORDINARY MEETING.

Monday, 5*th February* 1900.

The Rev. Professor Duns, Vice-President, in the Chair.

The following Communications were read :—

1. On a Thermostat electrically heated and regulated. By Dr John Gibson and Alan W. C. Menzies, M.A., B.Sc.
2. On the Law of Elastic Fatigue. By Dr W. Peddie. (*Abstract.*) p. 90.
3. On Magnetic Screening. By Dr C. G. Knott.
4. The Clark Cell *versus* the Cadmium Cell as a Standard of Electromotive Force. By John Henderson, Esq., D.Sc., A.I.E.E.
5. The Action of Silver Salts on Solution of Ammonium Persulphate. By Hugh Marshall, D.Sc. pp. 163–168.

Mr Thomas P. Watson, Sir Bhagvat Sinh Jee, G.C.I.E., H.H. the Thakore Sahib of Gondal, and Mr Douglas A. Gilchrist were balloted for, and declared duly elected Fellows of the Society.

SIXTH ORDINARY MEETING.

Monday, 19*th February* 1900.

Professor M'Kendrick, Vice-President, in the Chair.

Sir John Sibbald gave an address—

"On the Statistics of Suicide in Scotland."

SEVENTH ORDINARY MEETING.

Monday, 5*th March* 1900.

The Rev. Professor Duns, D.D., Vice-President, in the Chair.

The following Communications were read :—

1. On certain Aggregates of Determinant Minors. By Thomas Muir, Esq., LL.D. pp. 142–154.

2. Notes on the Dynamics of Cyclones. Part I. By JOHN AITKEN, Esq., F.R.S. *Trans.*, vol. 40, pp. 131–148.

3. Note on the Activity of Saliva in Diseased Conditions of the Body. By W. G. AITCHISON ROBERTSON, M.D., D.Sc. pp. 155–157.

The Society at this Meeting adopted the recommendation of the Council, intimated to the Society at the Fifth Ordinary Meeting on the 15th of February 1900, that the following changes be made in the Laws :—

"That Law XIV. read—The Ordinary Meetings shall be held on the "First and Third Mondays of each month from November to "March, and from May to July, inclusive ; with the exception "that when there are five Mondays in January, the Meetings "for that month shall be held on its Second and Fourth "Mondays."

"That Law XIX. read—An Extraordinary Meeting for the election "of Office-Bearers shall be held annually on the Fourth Monday "of October, or on such other lawful day in October as the "Council may fix, and each Session of the Society shall be held "to begin at the date of the said Extraordinary Meeting."

"In Laws XXI. and XXII. read October for November."

Mr DAVID SMILES JERDAN, Dr JOHN S. FLETT, Mr W. L. SARGANT, Mr T. EDGECUMBE EDWARDES, Prof. EDWARD ALBERT SCHÄFER, and Dr GEORGE ARCHDALL O'BRIEN REID were balloted for, and declared duly elected Fellows of the Society.

EIGHTH ORDINARY MEETING.

Monday, 19*th March* 1900.

The Right Hon. Lord Kelvin, G.C.V.O., President, in the Chair.

The following Communications were read :—

1. A Development of a Pfaffian having a Vacant Minor. By THOMAS MUIR, LL.D. *Trans.*, vol. 40, pp. 49–58.

2. The Theory of Alternants in the Historical Order of its Development up to 1841. By the Same. pp. 93–132.

3. Jacobi's Expansion for the Difference-Product, when the number of elements is even. By the Same. pp. 133–141.

4. Heat of Combination of Metals in the Formation of Alloys. By ALEXANDER GALT, D.Sc.

NINTH ORDINARY MEETING.

Monday, 7th May 1900.

Sir Arthur Mitchell, K.C.B., Vice-President, in the Chair.

Mr James Bisset and Mr Thomas P. Watson were admitted Fellows of the Society.

The following Communications were read:—

1. On the Dynamics of Cyclones and Anticyclones. Part II. By John Aitken, F.R.S. *Trans.*, vol. 40, pp. 148–152.

2. Observations on certain Nemerteans from Singapore. By R. C. Punnett, B.A. Communicated by A. T. Masterman, D.Sc. pp. 91–92.

3. The Reduction to Sea-Level of the Ben Nevis Barometer. By R. T. Omond.

Dr John Souttar M'Kendrick and Dr Joseph M'Gregor Robertson were balloted for, and declared duly elected Fellows of the Society.

TENTH ORDINARY MEETING.

Monday, 21st May 1900.

Mr A. Beatson Bell in the Chair.

Dr Burgess and Dr Traquair, the Representatives of the Society at the Bicentenary of the Royal Prussian Academy of Sciences, gave a brief account of the proceedings.

The following Communication was read:—

On *Tetrabothrium torulosum* and *Tetrabothrium auriculatum.* By Dr O. von Linstow, Göttingen. Communicated by Sir John Murray, K.C.B. pp. 158–160.

ELEVENTH ORDINARY MEETING.

Monday, 4th June 1900.

Professor M'Kendrick, M.D., Vice-President, in the Chair.

Dr John Souttar M'Kendrick was admitted a Fellow of the Society.

The following Communications were read:—

1. Studies in Coleopterous life-histories:—(*a*) On the Biology of *Pissodes pini.* pp. 319–358. (*b*) On the Biology of *Scolytus multistriatus.*

pp. 359–364. By R. STEWART MACDOUGALL, D.Sc. With illustrative examples of the Insects and their Work. Communicated by Professor COSSAR EWART.

2. On the Physical, Chemical, and Biological Conditions of the Black Sea. By Sir JOHN MURRAY, K.C.B.

J. M'LAUCHLAN YOUNG was balloted for, and declared duly elected a Fellow of the Society.

Dr EDWARD CAIRD, Master of Balliol College, Oxford; Dr DAVID FERRIER, Professor of Neuro-Pathology, King's College, London; Dr GEORGE FRANCIS FITZGERALD, Professor of Natural and Experimental Philosophy, Trinity College, Dublin; Dr ANDREW RUSSELL FORSYTH, Sadlerian Professor of Pure Mathematics in the University of Cambridge; Dr ARCHIBALD LIVERSIDGE, Professor of Chemistry in the University of Sydney; and Dr THOMAS EDWARD THORPE, Principal of the Government Laboratories, London, were balloted for, and declared duly elected British Honorary Fellows.

Dr ARTHUR AUWERS, Secretary, Royal Prussian Academy of Sciences; Professor WILHELM HIS, Leipzig; Professor ADOLF RITTER VON BAEYER, Munich, were balloted for, and declared duly elected Foreign Honorary Fellows.

TWELFTH ORDINARY MEETING.

Monday, 18th June 1900.

Dr James Burgess in the Chair.

The following Communications were read:—

1. The Total Solar Eclipse of 28th May 1900. By Mr THOMAS HEATH. pp. 236–247.

2. The Observations made at the Ben Nevis Observatories from 1883, and their publication. By Dr A. BUCHAN, F.R.S., and Mr R. T. OMOND.

THIRTEENTH ORDINARY MEETING.

Monday, 2nd July 1900.

Sir Arthur Mitchell, K.C.B., Vice-President, in the Chair.

Dr JOHN W. H. EYRE was admitted a Fellow of the Society.

The following Communications were read:—

1. On the Craniology of the People of India. Part II. By Prof.

Sir WILLIAM TURNER, F.R.S. *Trans.*, vol. 40, pp. 59–129. (*Abstract.*) pp. 161–162.

2. A Bathymetrical Survey of the Scottish Fresh-water Lochs: Lochs Chon, Ard, Menteith, Earn, Leven, Garry, and Ericht; with Observations on the Distribution of Temperature in the Water of these Lochs. By Sir JOHN MURRAY, K.C.B., and Mr FRED. P. PULLAR, F.R.G.S.

3. Further Note on the Preparation of the Diamond:—a Claim for Priority. By R. SYDNEY MARSDEN, M.B., D.Sc.

Mr JAMES YOUNG SIMPSON, Dr WILLIAM GAYTON, Mr JAMES MITCHELL, Mr JAMES BOWER BENNETT, and Dr NATHAN RAW were balloted for, and declared duly elected Fellows of the Society.

FOURTEENTH AND LAST ORDINARY MEETING.

Monday, 16th July 1900.

Professor Copeland, and, subsequently, The Right Hon. Lord Kelvin, G.C.V.O., President, in the Chair.

Mr JAMES MITCHELL was admitted a Fellow of the Society.

The following Communications were read:—

1. On the Motion produced in an Infinite Elastic Solid by the Motion, through the Space occupied by it, of a body acting on it only by Attraction or Repulsion. By the Rt. Hon. LORD KELVIN, President. pp. 218–235.

2. On the Number of Molecules in a cubic centimetre of Gas. By the Same.

3. Hyperbolic Quaternions. By ALEXANDER MACFARLANE, M.A., D.Sc. pp. 169–180.

4. Preliminary Note on the Deep-Sea Deposits collected during the "Valdivia" Expedition. By Sir JOHN MURRAY, K.C.B., and Dr E. PHILIPPI.

5. Leakage of Electricity from Charged Bodies at Moderate Temperatures. II. By Professor J. C. BEATTIE, D.Sc.

6. The Theory of Skew Determinants and Pfaffians in the historical order of its development up to 1857. By THOMAS MUIR, LL.D. pp. 181–217.

7. Brief Review of the Session. By the PRESIDENT.

Meetings of the Royal Society—Session 1900–1901.

THE 118TH SESSION.

GENERAL STATUTORY MEETING.

Monday, 22nd October 1900.

The following Council were elected :—

President.

THE RIGHT HON. LORD KELVIN, G.C.V.O., F.R.S.

Vice-Presidents.

Professor CHRYSTAL, LL.D.
Sir ARTHUR MITCHELL, K.C.B., LL.D.
Sir WILLIAM TURNER, M.B., F.R.S.
Professor COPELAND, Astronomer-Royal for Scotland.
The Rev. Professor DUNS, D.D.
Prof. JAMES GEIKIE, LL.D., F.R.S.

General Secretary—Professor P. G. TAIT.

Secretaries to Ordinary Meetings.

Professor CRUM BROWN, F.R.S.
RAMSAY H. TRAQUAIR, M.D., LL.D., F.R.S.

Treasurer—PHILIP R. D. MACLAGAN, Esq., F.F.A.

Curator of Library and Museum—ALEXANDER BUCHAN, Esq., M.A., LL.D., F.R.S.

Ordinary Members of Council.

The Hon. Lord M'LAREN, LL.D.
C. G. KNOTT, Esq., D.Sc.
Dr ALEX. BRUCE, M.A., F.R.C.P.E.
JAMES A. WENLEY, Esq.
The Rev. Professor FLINT, D.D.
JAMES BURGESS, Esq., C.I.E., LL.D.
R. M. FERGUSON, Esq., Ph.D., LL.D.
ROBERT IRVINE, Esq., F.C.S.
Professor JOHN G. M'KENDRICK, M.D., LL.D., F.R.S.
Professor SCHÄFER, F.R.S.
Dr ROBERT MUNRO, M.A.
J. S. MACKAY, Esq., LL.D.

FIRST ORDINARY MEETING.

Monday, 5th November 1900.

Sir Arthur Mitchell, K.C.B., Vice-President, in the Chair.

The Chairman, on opening the Session, made the following Statement:—

During the past Session 48 papers, many of them involving much ingenious research, have been communicated to the Society. Of these, 15 belong to the department of Physics, 9 to Mathematics, 3 to Chemistry, 2 to Astronomy, 3 to Oceanography, 5 to Biology, 1 to Human Anatomy, 2 to Comparative Anatomy, 3 to Physiology, 4 to Meteorology, and 1 to Social Statistics.

Since the commencement of the Session 23 Fellows have been added to our numbers. Of these, 2 are Professors, 6 are Lecturers on Science, 3 are Doctors of Science, 3 have the degree of M.D., and 2 that of LL.D.

I regret to say that during the same period the Society has lost by death 11 members, among whom are two of its Hon. Vice-Presidents, having formerly filled the office of President,—the Duke of Argyll and Sir Douglas Maclagan.

The Duke of Argyll, besides being President of this Society from 1860 to 1864, held at various times the offices of Chancellor of the University of St Andrews, Lord Rector of the University of Glasgow, and President of the British Association for the Advancement of Science in 1861. Theological controversy, metaphysical speculation, economical inquiries, historical research, and geology were subjects all ably treated of in his various publications, whilst as a statesman he initiated and supported much useful legislation.

Sir Douglas Maclagan was President of this Society from 1890 to 1894. He held in a distinguished manner for thirty-four years the Chair of Forensic Medicine, and was trusted adviser of the Crown in trials where forensic advice was required. His genial presence among us, now lost, is a happy memory to many of our Fellows.

Professor Sir T. GRAINGER STEWART held for twenty-three years the position of Professor of the Practice of Medicine, and he worthily maintained the high traditions of the Chair of Cullen, Gregory, and Alison.

Professor PIAZZI SMYTH published works on the Great Pyramid which have attracted much notice both in this country and the United States, and his spectroscopic studies and researches are of great cosmical interest.

ADAM GILLIES SMITH discharged with great acceptance the duties of Treasurer of the Society.

ROBERT HALLIDAY GUNNING, LL.D., Grand Dignitary of the Order of the Rose of Brazil, the munificent founder of the Prize which bears his name, will be long affectionately remembered for his genial and unassuming disposition, and for his many deeds of enlightened beneficence.

Dr JOHN ANDERSON, a native of this city, and a distinguished graduate of our University, was from 1865 to 1886 Superintendent of the Indian Museum at Calcutta. He was the author of several valuable works on the Vertebrata of India, Siam, Arabia, and Egypt.

DAVID BRUCE PEEBLES, an able Engineer.

PETER MACLAGAN of Pumpherston, formerly Member of Parliament for Linlithgowshire.

The Society having been invited to send two Delegates to represent it at the celebration of the Bicentenary of the Royal Prussian Academy, Dr BURGESS and Dr TRAQUAIR were appointed the Society's representatives. At a meeting of the Academy convened to receive the congratulations of the Delegates from Societies, the following Address was presented and read in the name of the Society:—

IN THE NAME, and by the authority, of the Council of the ROYAL SOCIETY OF EDINBURGH, we hereby offer our warmest congratulations to the ROYAL ACADEMY OF SCIENCES OF PRUSSIA on the attainment of its two hundredth anniversary.

We rejoice to recognise that the Royal Academy of Sciences of Prussia stands in the very front rank of the Learned Societies of the world. Alike in mathematics and physics, in history,

philology, and philosophy, it has, throughout almost its whole existence, counted among its members an extraordinary number of the most renowned and fruitful investigators. It has successfully carried on vast and erudite labours which have made all scholars its debtors, and stimulated numerous researches of great national and general utility.

This Society sincerely sympathises with the Royal Academy of Prussia in the losses which it has sustained in recent years through the deaths of VON HELMHOLTZ, of VON HOFMANN, of DU BOIS-REYMOND, of ERNST CURTIUS, of WAITZ, and of WATTENBACH, and other eminent and honoured members of the Academy; while it recalls with satisfaction that it has counted, and still counts, among its own Honorary Fellows, members of the illustrious Academy.

The Royal Society of Edinburgh hopes that the Academy may have continually increasing prosperity, and that all its labours may contribute to the glory of the German Empire and the enlightenment and progress of humanity.

In the name of the Royal Society of Edinburgh,

L.S. (Signed) KELVIN, *President.*
P. G. TAIT, *Secretary.*

March 9th, 1900.

Our Representatives reported that they were hospitably received, and had the honour of lunching with the German Emperor.

Mr CHARLES PIAZZI SMYTH has bequeathed a sum calculated to amount to about £10,000, to be ultimately administered by this Society, but in the meantime to be held in trust for certain beneficiaries, and subject to their life interest, and on the decease of these beneficiaries, the above mentioned sum to be held in trust by the Society, whereof the annual income is to be employed—(1) in printing, at a cost of about £600, his spectroscopic MSS.; and (2) in assisting or promoting, at an interval of every ten or twenty years, an exceptional expedition for the study of some particular branch of astronomical spectroscopy in the purer air of

some mountain elevations of not less than 6000 feet above the sea-level, as tried and found feasible by him in a first experiment on the Peak of Teneriffe. The testator also bequeaths to the Society all his books of original drawings and journals, and all his boxes of glass photographs, and likewise his portrait by Mr Faed, R.S.A.

It was announced in the *Times* of 31st May of this year that the Government had appointed a Committee, *inter alia*, for suggesting changes in the staff and arrangements necessary for bringing the Geological Survey in its more general features to a speedy and satisfactory termination, and in connection with this, the following representation was submitted to the Committee on the part of the Society:—

"The Council, in the interest alike of science and of the industrial or economic development of the country, wishes to express its conviction that no termination of the Survey will be considered satisfactory in Scotland unless the survey of the country is completed on the 6 inch scale, and its hope that, whatever arrangements the Committee may recommend, this specially important point will be kept in view.

"Should it be desired by the Committee, the Council is prepared to send representatives to give evidence regarding the future work of the Geological Survey in Scotland."

The President and Council of the Royal Society of London have made a Grant towards meeting the expense of publishing the Observations made at the Ben Nevis Observatory of a sum corresponding to the half of the whole expenditure expected to be incurred. The half will amount to £500. The Royal Society of Edinburgh will pay the other half.

Dr Copeland, Astronomer-Royal for Scotland, and his Assistant, Mr Heath, proceeded to Santa Pola in Spain, with suitable apparatus, to observe the eclipse of the sun.

The Society continues to take a great interest in Antarctic Exploration. A British Expedition will sail next year for the exploration of that part of the Antarctic Continent which lies south of the Pacific Ocean, and a German Expedition will explore that part of the Continent which lies south of the Indian Ocean. But expeditions limited to the investigation of these regions will leave a considerable part of the great South Polar Continent unexplored.

It has, therefore, been proposed that a Scottish Expedition should be organised to supplement the work of the British and German Expeditions. It would undertake the exploration of that part of the Antarctic Continent which lies south of South America. It is calculated that £35,000 would be required to provide a suitable vessel, with the necessary equipments of men, instruments, provisions, etc., for the purpose. Of this sum £10,000 have been promised.

The proposed staff includes six scientific men, five ship's officers, and a crew of twenty-six. The scientists will take systematic observations both on land and sea in meteorology, magnetism, terrestrial physics, biology, geology, hydrography, and other branches of inquiry. The Expedition would be under the command of Mr William S. Bruce, who has had great experience in Polar expeditions, having been five summers and one winter in the Polar regions, where he distinguished himself as an Arctic zoologist, having brought back larger zoological collections than any of his predecessors.

The following Communications were read :—

1. Dietary Studies of the Poorer Classes. By Dr Noël Paton, Dr J. C. Dunlop, and Dr Elsie Inglis.

2. Note on the Relations amongst the Thermo- and Electro-Magnetic Effects. By W. Peddie, D.Sc.

SECOND ORDINARY MEETING.

Monday, 19*th November* 1900.

The Astronomer-Royal for Scotland, Vice-President, in the Chair.

The Chairman gave the substance of Communications from the Scottish Office, Whitehall, and from the Nobel Committee of the Royal Swedish Academy of Sciences, as to the Nobel Foundation.

The following Communications were read :—

1. Diurnal Range of Temperature in the Mediterranean during the Summer Months. By Alexander Buchan, LL.D., F.R.S.

2. The Topography of the Gray Matter and Motor Cell in the Spinal Cord. By Alexander Bruce, M.D.

A Ballot was held for the election of Dr ALEXANDER BUCHAN, who had been nominated by the Council to succeed Sir JOHN MURRAY as the Society's Representative on the Heriot-Watt Trust, and Dr BUCHAN was duly elected.

THIRD ORDINARY MEETING.

Monday, 3rd December 1900.

The Rev. Professor Duns, D.D., Vice-President, in the Chair.

The following Communications were read :—

1. The True Cæcal Apex, or the Vermiform Appendix—its Minute and Comparative Anatomy. By RICHARD J. A. BERRY, M.D. (*With Lantern Illustrations.*)
2. Some Identities connected with Alternants, and with Elliptic Functions. By THOMAS MUIR, LL.D. *Trans.* vol. 40, pp. 187–201.
3. A Peculiar Set of Linear Equations. By the Same. pp. 248–260.

Mr ALAN W. C. MENZIES and Professor J. B. BRADBURY were balloted for, and declared duly elected Fellows of the Society.

FOURTH ORDINARY MEETING.

Monday, 17th December 1900.

The Right Hon. Lord Kelvin, G.C.V.O., President, in the Chair.

Mr ALAN W. C. MENZIES was admitted a Fellow of the Society.

The following Communications were read :—

1. On the Transmission of Force. By the PRESIDENT.
2. Note on Dr Muir's paper "On a Peculiar Set of Linear Equations." By C. TWEEDIE, Esq., M.A. pp. 261–263.
3. A Suggested Solar Oscillation, with some of its possible Astronomical and Meteorological consequences; together with a Generalisation as to the Constitution of Matter and the Cause of Gravitation. By Professor J. T. MORRISON.

FIFTH ORDINARY MEETING.

Monday, 7th January 1901.

The Astronomer-Royal for Scotland, Vice-President, in the Chair.

Mr J. B. Bennett was admitted a Fellow of the Society.

The following Communication was read:—

Exploration in Spitzbergen, and Soundings in Seas adjacent, in 1898 and 1899. By William S. Bruce, Esq. Communicated by Dr Buchan. (*With Limelight Illustrations.*)

Mr Fred. P. Pullar, Dr Carstairs Cumming Douglas, and Dr R. Stewart MacDougall were balloted for, and declared duly elected Fellows of the Society.

SIXTH ORDINARY MEETING.

Monday, 21st January 1901.

The Right Hon. Lord Kelvin, G.C.V.O., President, in the Chair.

The following Communications were read:—

1. One-dimensional Illustrations of the Kinetic Theory of Gases. By the Chairman.
2. Note on Solar Radiation and Earth Temperatures. By Professor Knott, D.Sc. pp. 296–311.
3. Note on Pairs of Consecutive Integers, the Sum of whose Squares is an Integral Square. By Thomas Muir, Esq., LL.D. pp. 264–267.
4. The Differentiation of a Continuant. By Thomas Muir, Esq., LL.D. *Trans.*, vol. 40, pp. 209–220.
5. The Hessian of a General Determinant. By Thomas Muir, Esq., LL.D. *Trans.*, vol. 40, pp. 203–207.

SEVENTH ORDINARY MEETING.

Monday, 4th February 1901.

Professor Chrystal, LL.D., Vice-President, in the Chair.

Dr Carstairs C. Douglas was admitted a Fellow of the Society.

The Chairman read the following Address which had been presented to His Majesty King Edward on his accession to the Throne:—

To the King's Most Excellent Majesty, the Loyal and Dutiful Address of the Royal Society of Edinburgh.

May it please Your Majesty:

We, the Royal Society of Edinburgh, humbly approach Your Majesty, on your accession to the Throne, with the expression of our sincere and earnest sympathy towards yourself, your Royal Consort, and the members of the Royal Family, on your bereavement, and our sense of the great loss which has befallen the nation through the death of our revered and beloved Sovereign, Queen Victoria.

We feel assured that the memory of your Royal Mother, the late Queen, whose life was devoted to the welfare of her subjects, will ever be held in affectionate remembrance by all who are privileged to owe allegiance to the Crown, and that Her Majesty's name will be illustrious in history, not only for the greatness and power of the Empire which was consolidated in her reign, but for the wisdom and justice with which the Empire was administered under her guidance and example.

We desire respectfully to express our good wishes and our hope that Your Majesty may enjoy a long and prosperous reign, as Sovereign of the many territories and races over which you have been called by Divine providence to preside. Your Majesty's most gracious assurance that your life would be devoted to the service of the State, springs from the same sense of public duty which inspired our lamented Queen, and gives the promise of a brilliant and prosperous future for the Empire under your Majesty's sovereignty, which we trust may be of long duration.

Following the example and inclination of your revered Father, the Prince Consort, Your Majesty has shown a warm interest in the advancement of science, literature, and art; and we feel sure that it will be in accordance with Your Majesty's feelings and wishes that your reign may be distinguished by the progress of the nation in all fields of intellectual activity.

We ask permission also to offer to Her Gracious Majesty the Queen Consort our respectful good wishes on her accession to the great position for which she is so eminently qualified.

In the name of the Royal Society of Edinburgh,

L.S. (Signed) KELVIN, *President.*
JOHN M'LAREN, *Acting Secretary.*

January 25*th*, 1901.

The following Communications were read:—

1. Obituary Notice of His Excellency Dr Gunning. By Professor DUNS, D.D., Vice-President. pp. 489–497.

2. Solar Radiation and Earth Temperatures. Part II. By C. G. KNOTT, D.Sc. pp. 296–311.

EIGHTH ORDINARY MEETING.

Monday, 18*th February* 1901.

Professor Geikie, LL.D., Vice-President, in the Chair.

The following Communications were read:—

1. Thermo-electric Properties of Solid Mercury. By Dr W. PEDDIE and the late Mr A. B. SHAND. p. 422.

2. Observations of the Edinburgh Rock Thermometers. By THOMAS HEATH, Esq., B.A. *Trans.*, vol. 40, pp. 157–186.

NINTH ORDINARY MEETING.

Monday, 4*th March* 1901.

Sir Arthur Mitchell, K.C.B., Vice-President, in the Chair.

Mr JAMES YOUNG SIMPSON was admitted a Fellow of the Society.

The following Communications were read:—

1. The Sea-weed *Ulva latissima*, and its relation to the Pollution of Sea-water by Sewage. By Professor LETTS and Mr JOHN HAWTHORNE, B.A., Queen's College, Belfast. pp. 268–294.

2. Further Notes on the Dynamics of Cyclones and Anticyclones. By JOHN AITKEN, Esq., F.R.S. *Trans.*, vol. 40, pp. 152–156.

3. Note on the New Star in Perseus. By the ASTRONOMER-ROYAL FOR SCOTLAND. pp. 365–369.

Mr F. H. A. MARSHALL was balloted for, and declared duly elected a Fellow of the Society.

TENTH ORDINARY MEETING.

Monday, 18th March 1901.

Professor Geikie, LL.D., Vice-President, in the Chair.

Dr JOHN S. FLETT was admitted a Fellow of the Society.

The following Communications were read:—

1. The Old Red Sandstone of Shetland, and its relation to the Old Red Sandstone of the rest of Scotland. By JOHN S. FLETT, M.A., D.Sc. (*With Lantern Illustrations.*)

2. On Fossil Fishes collected by Dr Flett in the Old Red Sandstone of Shetland. By Dr R. H. TRAQUAIR, F.R.S. (*With Lantern Illustrations.*)

3. On Dipnoi from the Upper Old Red Sandstone of Scotland. By Dr R. H. TRAQUAIR, F.R.S. (*With Lantern Illustrations.*)

ELEVENTH ORDINARY MEETING.

Monday, 6th May 1901.

Dr James Burgess in the Chair.

The Chairman read the reply which His Majesty the King had been graciously pleased to send, through the Secretary for Scotland, to the President, in answer to the recent Address of Condolence and Congratulation of the Society.

The following Communications were read:—

1. Further Notes on the New Star in Perseus. By the ASTRONOMER-ROYAL FOR SCOTLAND and Dr J. HALM.

2. On Certain Relations between the Electrical Conductivity and the Chemical Character of Solutions. By Dr JOHN GIBSON.

3. Additional Note on the Ultra-Neptunian Planet whose Existence is indicated by its Action on Comets. By Professor GEORGE FORBES, F.R.S. pp. 370–374.

Dr W. BRODIE BRODIE, Dr H. S. CARSLAW, Mr THOMAS W. DRINKWATER, Prof. SANJIBAN GANGULI, Dr DAVID WATERSTON, and Mr JAMES MORE, jun., were balloted for, and declared duly elected Fellows of the Society.

TWELFTH ORDINARY MEETING.

Monday, 20th May 1901.

Professor Geikie, LL.D., Vice-President, in the Chair.

The following Communication was read:—

Ice-Erosion in the Cuillin Hills, Skye. By ALFRED HARKER, Esq., M.A., F.G.S., H.M. Geological Survey of Scotland. Communicated by JOHN HORNE, Esq., F.R.S. *Trans.*, vol. 40, pp. 221–252.

THIRTEENTH ORDINARY MEETING.

Monday, 3rd June 1901.

Dr David Hepburn in the Chair.

Mr ARCHDALL REID, M.B., Mr F. H. A. MARSHALL, and Dr DAVID WATERSTON were admitted Fellows of the Society.

The following Communications were read:—

1. Observations on Binary Fission in the Life-History of Ciliata. By Dr J. Y. SIMPSON. pp. 401–421.

2. Apparatus for Measuring Strain and Applying Stress. By E. G. COKER, Esq., D.Sc. Communicated by Dr C. G. KNOTT. *Trans.*, vol. 40, pp. 263–294.

3. On the Anatomy of a Collection of Slugs from N.W. Borneo. By WALTER E. COLLINGE, Esq. Communicated by Prof. W. C. M'INTOSH. *Trans.*, vol. 40, pp. 295–312.

Dr ROBERT JARDINE and Mr EDWARD SMART were balloted for, and declared duly elected Fellows of the Society.

FOURTEENTH ORDINARY MEETING.

Monday, 17th June 1901.

Professor Sir Wm. Turner, K.C.B., Vice-President, in the Chair.

The following Communications were read :—

1. On In-breeding. By Professor J. COSSAR EWART, F.R.S.
2. On Hair in the Equidæ. By F. H. A. MARSHALL, Esq., B.A. pp. 375–390.

FIFTEENTH ORDINARY MEETING.

Monday, 1st July 1901.

Professor Chrystal, LL.D., Vice-President, in the Chair.

Dr W. BRODIE BRODIE was admitted a Fellow of the Society.

The following Communications were read :—

1. Note on a Proposition given by Jacobi in his "De determinantibus functionalibus." By THOMAS MUIR, Esq., LL.D. pp. 423–427.
2. On the Distribution of Fossil Fishes in the Carboniferous Rocks of the Edinburgh District. By Dr R. H. TRAQUAIR, F.R.S.
3. The Determination of Sex in Animal Development. By J. BEARD, D.Sc. Communicated by Prof. COSSAR EWART, F.R.S.

Mr JAMES GOODWILLIE, the Rev. G. A. FRANK KNIGHT, Dr O. ST JOHN MOSES, and Mr DAVID PATERSON were balloted for, and declared duly elected Fellows of the Society.

SIXTEENTH AND LAST ORDINARY MEETING.

Monday, 15th July 1901.

The Rev. Professor Flint, D.D., in the Chair.

The Chairman referred in a few appropriate words to the great loss which the Society had sustained by the death of Professor TAIT.

The Gunning Victoria Jubilee Prize for 1896–1900 was presented

to Dr T. D. Anderson for his discoveries of New and Variable Stars.

The Chairman, on presenting the Prize, said :—

The Council of the Royal Society of Edinburgh have decided to award the Gunning Prize to Dr T. D. Anderson for his distinguished services to astronomical science. Dr Anderson's name has come prominently before the astronomical world by his discovery of a large number of variable stars, visible in our latitudes, as well as of two temporary stars, one in the constellation of Auriga and the other in that of Perseus. In the present highly developed state of stellar spectroscopy, the discovery of these two remarkable stars in such close succession was bound to lead to a considerable enrichment of our knowledge with regard to the physical constitution of these celestial bodies, and still promises to shed new light on important and perplexing problems in the domain of stellar evolution. In the case of Nova Persei, the present new star, the value of Dr Anderson's timely discovery is enhanced by the fact that it afforded astronomers the unique opportunity for watching the course of development in the *initial* stages of this phenomenon, and in this respect the importance of the discovery has been fully appreciated by astro-physicists.

Brilliant, however, as these startling discoveries undoubtedly were, they are only, so to speak, incidental results of a lifelong labour devoted to a systematic search for variable stars; and this, indeed, is what constitutes Dr Anderson's principal contribution to astronomical science. The indomitable zeal and perseverance by which he has been enabled to add as many as thirty-five variables to the catalogue of this important class of celestial objects are all the more creditable to him, as the small optical power of the instruments at his disposal, and the distinctly unfavourable site of his private observatory, were bound to render his observations very difficult and laborious. Not being in possession of star-maps, the essential requirements for a work of this kind, Dr Anderson had to prepare his own charts from the star-catalogues of the Bonn Durchmusterung. The extremely fatiguing labour involved in the construction of these charts, which include more than 70,000 stars down to the 9·5th magnitude, is a signal proof of his enthusiastic

devotion to this particular branch of astronomical observati
is the desire of the Society to recognise by this award the va
importance of Dr Anderson's work in a field of astronom
search where results can be obtained only by the most dete
perseverance and by an unabating enthusiasm and love for

In conclusion, I have to express the extreme regret
Astronomer-Royal for Scotland that illness prevents hi
being present on this memorable occasion.

The Keith Prize for 1897–99 was presented to Dr
BURGESS for his paper "On the Definite Integral $\frac{2}{\sqrt{\pi}}$
with extended Tables of Values," printed in vol. xxxix.
Transactions of the Society.

The Chairman, on presenting the Prize, said :—

The Keith Prize for the Sessions 1897–8, 1898–9, is aw
James Burgess, Esq., C.I.E., LL.D., for his paper o
Definite Integral $\frac{2}{\sqrt{\pi}}\int_0^t \epsilon^{-t^2} dt$, with Extended Tables of
published in the Society's *Transactions*. This integral is o
ance in various fields of physical science, such as theory of atm
refraction, conduction of heat, probabilities, errors of obs
etc. It is also of fundamental importance in the evalu
many other forms of definite integrals. A closely c
integral was tabulated in 1789 by Kramp, and various
both integrals have been computed or compiled by
authors since that date. Dr Burgess's tables are, howeve
lated to a greater number of significant figures than in any
earlier tables, being for certain values of the limit com
fifteen decimal places. The logarithms are in these cases
sixteen places, and the table is prepared for all practical
by being provided with tables of differences as far as th
order. The arithmetical labour involved in constructin
table must have been enormous, and could have been acco
only by a calculator of rare accuracy and power. In ad
the tabulated values, which fill thirty-nine pages of the
Transactions, the memoir itself contains a brief histor
subject, and a luminous account of the methods adopted i

observation. It
rd the value and
astronomical re-
nost determined
love for science.
e regret of the
vents him from

to Dr JAMES
gral $\frac{2}{\sqrt{\pi}}\int_0^t \epsilon^{-\ ^2} dt$,
. xxxix. of the

9, is awarded to
aper on "The
les of Values,"
gral is of import-
y of atmospheric
of observation,
e evaluation of
sely connected
arious tables of
d by different
however, calcu-
in any of these
it computed to
e cases given to
actical purposes
r as the fourth
tructing such a
n accomplished
In addition to
of the Society's
history of the
pted in making

and in checking the calculations. The section on Interpolation in particular, a valuable addition to mathematical literature, shows that the author is as well fitted to extend mathemat theory as to compute mathematical constants to thirty signific figures. In awarding Dr Burgess the Keith Prize, the Cou have considered the pure mathematical interest of the proce involved, as well as the great practical value of this admirable finished piece of work.

The Makdougall-Brisbane Prize for 1898–1900 was presente Dr RAMSAY H. TRAQUAIR for his paper entitled "Report on F Fishes collected by the Geological Survey in the Upper Silu Rocks of Scotland," printed in vol. xxxix. of the *Transaction* the Society.

The Chairman, on presenting the Prize, said:—

Dr Traquair's report on the Fossil Fishes discovered by Geological Survey in the Upper Silurian Rocks of Scot furnishes striking proof of his thorough knowledge of Palæ Ichthyology. His researches have proved of exceptional value a biological point of view. By means of these fossils, all of w are new to science, he has advanced a new classification of *Ostracodermi*, which now comprises three orders. He has enla our knowledge of the order *Heterostraci*, which now includes families instead of one. He has shown that the *Cœlolepidæ*, th probably of Elasmobranch origin, were not Cestraciont sharks, has indicated the transition from the *Cœlolepidæ* to the *Pterasp* These are only some of the important features of his researches results of which have been of the highest value on account of light which they throw on the evolution of these Palæozoic fis

The following Communications were read:—

1. The General Form of the Involutive 1–1 Quadric Transform in a Plane. By CHARLES TWEEDIE, M.A., B.Sc. *Trans.*, vol. 40, 253–262.

2. Supplementary Report on Fossil Fishes collected by the Geolo Survey in the Silurian Rocks of the South of Scotland. By Dr TRAQUAIR, F.R.S.

3. Exhibition of Photographs of the Corona taken during the Eclipse of 28th May 1900. By THOMAS HEATH, B.A. pp. 396–40

4. The true Shape, Relation, and Structure of the Alimentary Viscera of the Common Porpoise (*Phocœna communis*), as displayed by the Formal Method. By David Hepburn, M.D., and David Waterston, M.A., M.D. (*With Lantern Illustrations.*) *Trans.*, vol. 40.

5. On the Central Plexus of *Cephalodiscus dodecalophus*, M'I. By A. T. Masterman, M.A., D.Sc.

6. By permission of the Society, a paper entitled "Notes on the Appearance of some Foraminifera in the Living Condition," by Frederick Chapman, A.L.S., F.R.M.S., and communicated by Sir John Murray, K.C.B., was laid on the table. pp. 391–395.

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OBITUARY NOTICES.

His Excellency R. H. Gunning, Esq., M.D., LL.D., etc.

By **Professor Duns**, D.D., Vice-President.

(Read February 4, 1901.)

I need hardly remind the Society that, at the first meeting of the Session, the Chairman is expected to refer to the Fellows who have died in the course of the year. In a few words mention was made of the death of His Excellency Robert Halliday Gunning, Esq., M.D., LL.D., F.S.A. Scot., and the Vice-President who occupied the Chair intimated that I would prepare a fuller notice of His Excellency later on. When looking at Dr Gunning's relation to this Society it is worth noting that the Fellows consist of five classes:—(1) those who join it with the intention of contributing to its literature; (2) those who listen with pleasure to the things new and old which the Proceedings reveal; (3) those who find in the title F.R.S.E. an honour and, in many cases, a true help in their life's work; (4) those who set a high value on the work done by the Society, who in the past have been, and no doubt in the future will be, helpful by money endowments; and (5) Honorary Fellows—men of this and other lands who are celebrated by original contributions to one branch or to more than one branch of science. Numbers 2 and 4 are specially represented, both in the motive and the method of true science, by the personal friend of whom I now write.

When Napoleon heard any one praised highly he was wont to ask, "What has he done?" Is this relevant in the present case? I think it is, though the proofs of Dr Gunning's '*doing*' often come, not in scientific sequence, but are frequently suggestive of missing links. Anticipation becomes mixed with retrospect and the association is mutually interesting. Both testify to a busy life. In a letter to me, so recently as August 1899, we have a good illustration of his frequently linking the chief events of his

changeful life with matters which might have stood alone, whose connection, however, gave them a place of importance which they could not otherwise have had. The mention of a comparatively small matter leads him to think of his childhood, and then to hasten to dwell on the upward steps of his experience. I notice this in answer to the query, "What has he done?" It gives me the opportunity early in this sketch of bringing to the front his standing as a worker. "I am anxious," he says, "to determine some points about my family history. My mother belonged to the Dicksons of Gateside and Bankhead, and having lost both her parents in Dumfries when about nine years of age, she was taken to Gateside and brought up by her uncle, the laird. I was born in Ruthwell, 1818, but left, when only two or three years old, for Kirkbean, and afterwards Newabbey and Dumfries, whence I left for Edinburgh in 1834. My last visit to Dumfries and Newabbey was in 1839 and in 1839–40, and 1840–41 I went to Aberdeen as Assistant and Demonstrator of Anatomy to Dr Allen Thomson at Marischal College. I returned with him to Edinburgh in 1841–42, and when he was appointed to the Chair of Physiology I took charge of the Anatomical Rooms under Monro *tertius*, and afterwards lectured on anatomy in Surgeon's Square, and prepared a numerous class of students and graduates from all parts of the Empire for taking the Degree of M.D. in Scotland and the membership of Surgeon in London. In 1847 I was married, and in 1849 I was obliged to seek a warmer climate on account of my health. The great improvement of my health in Brazil, and the prospect of easy and lucrative medical practice, induced me to remain there for thirty-three years; and from the time of my return to England in 1882 on to 1896 I had never been to my native place; that is, I had been away from it between seventy and eighty years. In 1896 I took Lady Hughes [Mrs Gunning] to Dumfries, to show her my native haunts, and we drove by way of Glencaple and Bankhead to Ruthwell and returned to Dumfries. Blindness deprived me of seeing these various places. It was in connection with this visit that I thought I should do some little thing for my native place, as I had done for the neighbouring parish, Ecclefechan, in honour of Carlyle. My chief benefactions have been for Edinburgh, where I spent many happy days, but I

felt I should also remember my birthplace and Newabbey, where I was at school for some years before going to Edinburgh."

I am indebted to Dr Gunning's agents, Messrs Auld & Macdonald, W.S., for the following record of his chief benefactions:—The University of Edinburgh for Medical Prizes, £5000; the University of Edinburgh for Divinity Prizes, £5000; Protestant Institute of Scotland, £1000; Waldensian Missions Aid Society, £2500; Reformed Church of Bohemia, £2500; Evangelical Church of Italy, £2500; Royal Society of Edinburgh, £1000; Society of Antiquaries of Scotland, £1000; Association for University Education of Women, £1000; New College, Edinburgh, for Science Prizes, £1000; Royal College of Surgeons, Edinburgh, £1000; Royal College of Physicians, Edinburgh, £1000; Royal Society, London, £1000; Victoria Institute, London, £500; Dumfries Infirmary, £1250; and Robertson's Orphanage, South Queensferry, £1000; more than £28,000.

In forwarding this list Mr Macdonald adds:—"I enclose a list of Dr Gunning's benefactions which are passing through my hands. Of course his benefactions to the West Port Church, from first to last, must have come to a very large sum. He continued his subscriptions to it all the time he was in Brazil."

Now I am far from gauging the worth of a man by his wealth, or his greatness by his giving. But it seemed to me the only way to shed light on the individualism of one whose environments were often so many, and their influence on his every-day life so well marked. The list of his benefactions make it clear that he had determined to devote his riches only to schemes which were great and good. These considerations lead us to seek for links between his personal motives and every-day practices. The Institutions to the help of which his gifts were so generous were associated with philanthropy or with physical and natural science.

In looking over the material for this biographical notice, I am struck with Dr Gunning's frequent references to two men who, in their several departments of thought, were in their day men of mark, men appreciated by him while they lived and not forgotten after their death. One could not be long in his company without hearing him refer to one or the other—Thomas Chalmers, D.D., and Robert Christison, M.D. In this connection we find a key to

many things in Gunning's life: Chalmers the leader of theological thought and action; Christison the distinguished physician, well known as a man of high attainments, not only in medicine but in sciences outside of his personal profession. The influence of this acquaintanceship was the strengthening of those desires and ambitions which characterised and gave direction to Gunning's earnest efforts as indicated in the list of his benefactions. In making this statement I wish simply to show that Gunning admired both because he found in each elements with which he was in deep sympathy, and which would be helpful to himself in carrying out aspects of work which he loved and early began to take a lively interest in. In Mr Macdonald's communication a striking contrast is suggested without any break:—"Dr Gunning's interest," he says, "in Home Missions was aroused by Dr Chalmers, and he was one of the first elders ordained in the West Port, and Dr Gunning was created a Grand Dignitary of the Empire of Brazil by the Emperor Dom Pedro II., and this carried with it the right to be addressed as 'His Excellency.' The Emperor, a short time before his own expulsion from Brazil, wrote a holograph letter to the Queen asking that Dr Gunning should be authorised to use the rank in this country. The Queen granted this request, and Dr Gunning had a letter from Lord Salisbury intimating the fact."

The mission work was a great success, and His Excellency lived to take a leading part in laying the memorial stone of the present West Port Church, which has a congregation almost as large as the largest in Edinburgh.

Reference has been made to Sir Robert Christison as a friend of His Excellency, and helpful to him in trying to influence the Church in other than purely religious work. Chalmers had seen good opportunities for ministers benefiting society if, to their theological acquirements and teaching, they brought to their work the knowledge of one or more branches of physical or of natural science. In 1843 he had given great prominence to his views on this matter:—"We hold," he wrote, "a natural science class in connection with theology to be most desirable as a component part of our system of theological education." In this quotation I keep clear of seeming to discuss the question on the merits. I only wish to indicate the lines of public thought which led Dr Gunning

to devote large sums of money in its behalf. Chalmers, whose views impressed Gunning very much, was well acquainted with the apologetic value of such questions, and was in the habit of complaining that no provision was made in the theological course for it. There *might* be willing students, and Gunning resolved to do something for them. His strong efforts in this direction comes out in his correspondence with Sir Robert Christison. Sir Robert entered cordially into his proposals and brought them under the notice of leading University friends. The second object in the benefaction list, £5000, must be associated with Sir Robert Christison's friendly desires to help him to realise his long-cherished designs. I am greatly indebted to David Christison, Esq., M.D., for documents bearing on this and other matters. He says:—"I send you all the correspondence with Dr Gunning which my father had preserved. It relates, 1st, to the procuring of specimens of the ipecacuanha plant with the object of cultivating it in India, at a time when its enormous importance as a specific in dysentery, taken in large doses, was being realised. The 2nd series relates to the negotiations about the Gunning Fellowship." There are also documents bearing on Sir Robert's first acquaintance with him. Among the letters is one in which he informs Sir Robert that "Professor Agassiz passed a couple of days with him, seeking specimens of fresh-water fishes in the river not far from his residence. He was going south with Count Portales on the Gulph Stream Exploration." Gunning's mind was at the time charged with strong dislike of what he believed to be the tendency of the science of the day: "Telling Agassiz my disgust with the modern caricature of the doctrine of the *production* (spontaneous generation) and *reproduction* (evolution and development) of living beings, he thought well of my idea to help research for the solution of these questions." Another letter to Sir Robert is from Principal Tulloch, St Andrews, approving of his suggestions in favour of Dr Gunning's plans, and concluding:—"I do not think, therefore, you could give your friend better advice than what you indicated to me."

In the *Life of Sir Robert Christison* (vol. ii. p. 257) an extract from his private Journal (June 27, 1870) is given relating to ipecacuanha as referred to above. "A box of ipecacuanha plants arrived from Dr Gunning of Rio Janeiro. . . . . It has recently

been ascertained in China and India that it is a sovereign remedy for dysentery." It was a native of S. America, and Sir Robert had pressed for several years on his students the importance of introducing it into India. "Some months ago," he says, "I wrote to Dr Gunning, an Edinburgh graduate, who entered very cordially into the scheme. The first consignment of plants has just arrived at the Botanic Garden, consisting of roots well preserved in soil. . . . . I have seen to-day in the garden stove-house a hundred thriving young plants." Soon arrangements were made for introducing it into India, and he records that "there is a promise of four hundred more from the cuttings of Dr Gunning's consignment." I believe that ipecacuanha is still reared in India, and is regarded as a specific in dysentery. Be this as it may, it says much for Dr Gunning's zeal in his profession. Indeed the desire to work in its behalf led to that habit of the eye which characterised him until blindness overtook him, as it had done his father. One could not spend an hour with him without his varied scientific attainments coming to the front. The scientific references to Brazil were many and valuable, but he had also been a skilled observer in the home field. The fluviatile and glacial markings of his native district, and its zoology and antiquities, had occupied much of his attention in his student life. The so-called 'pots and pans' proofs of fluviatile action in the Kirkbean stream's course, or the history of the Ruthwell Stone, with its form and runes, and the value of its verses, were favourite themes.

There are many other facts which might be stated illustrative of His Excellency's Christian efforts, philanthropic movements, and friendly correspondence with members of the Royal families of Brazil and Portugal, which might be referred to here; but to dwell on these would be outside of the Society's intentions in this "Obituary Notice." I may, however, hark back for a little on the benefactions, and specially the "Jubilee Prizes," which pass into classes that will keep the occasion of their institution ever in remembrance, though to-day it is not the sound of the Jubilee trumpet but the wailing of the funeral dirge which fills men's ears and touches their hearts.* "The Gunning Victoria Jubilee Prize" was founded in 1887 by Dr R. H. Gunning, and is awarded

* Written on the day of HER MAJESTY's Funeral.

triennially by the Council of the Royal Society of Edinburgh, in recognition of original work in Physics, Chemistry, or pure or applied Mathematics. Evidence of such work may be afforded either by a paper on one of the above subjects, or some discovery in them, elsewhere communicated or made, which the Council may consider to be deserving of the prize. The prize consists of a sum of money, and is open to men of science resident in or connected with Scotland. The first award was made in the year 1887. In accordance with the wish of the donor, the Council of the Society may on fit occasions award the prize for work of a definite kind to be undertaken during the three succeeding years by a scientific man of recognised ability.

At the close of the first triennial period, 1884–87, the prize was awarded to Sir William Thomson, Pres. R.S.E., F.R.S. (Lord Kelvin), for a remarkable series of papers on "Hydrokinetics," especially on waves and vortices, which have been communicated to the Society. At the close of the second triennial period, 1887–90, it was awarded to Professor P. G. Tait, Sec. R.S.E., for his work in connection with the "Challenger" Expedition and his other researches in Physical Science. At the close of the third triennial period, 1890–93, it was awarded to Alexander Buchan, LL.D., for his varied, extensive, and extremely important contributions to Meteorology, many of which have appeared in the Society's publications. The last triennial award, 1893–96, was made to John Aitken, Esq., for his brilliant investigations in Physics, especially in connection with the Formation and Condensation of Aqueous Vapour.

The Gunning Fellowship in connection with the Society of Antiquaries of Scotland, constituted by the Victoria Jubilee gift of His Excellency Dr R. H. Gunning, "to enable experts to visit other museums, collections, or materials of archæological science at home or abroad, for purposes of special investigation and research," was inaugurated in the Jubilee year, 1887–88, by the appointment of Dr Joseph Anderson and Mr George F. Black to visit and report on local museums in Scotland. The Report, which extends to 160 pages, is printed, with illustrations, in the *Proceedings of the Society*, vol. xxii. p. 331. Under this Jubilee Gift the following appointments and additions have been made :—

In 1889 Dr Anderson was appointed to visit the museums of Switzerland and North Italy. His Report, extending to 32 pages, is printed in the *Proceedings*, vol. xxiv. p. 478.

In 1890–91 Mr J. Romilly Allen was appointed for two years to visit and report on the Sculptured Stones of Scotland, with a view to obtaining an archæological survey and description, with photographs, rubbings, or drawings of these monuments, for a work on the Early Christian Monuments of Scotland, to be issued by the Society. His first Report, "A Preliminary List of the Sculptured Stones of Scotland," is printed in the *Proceedings*, vol. xxiv. p. 510.

His second Report, "On the Sculptured Stones older than A.D. 1100, with Symbols and Celtic Ornament, in the district of Scotland north of the River Dee," is published in the *Proceedings*, vol. xxv. p. 422.

In 1892 Mr George F. Black was appointed to visit and report on the antiquities of the Culbin Sands, Morayshire. His Report, with numerous illustrations, is printed in the *Proceedings*, vol. xxv. p. 484.

In 1893 Mr George F. Black was appointed to visit and report on the Scottish Antiquities preserved in the British Museum, and the Museums of S. Kensington, the Society of Antiquaries, the Guildhall, and the Tower of London, and in the Museum of Science and Art, Edinburgh. His Report, with illustrations, is printed in the *Proceedings*, vol. xxvii. p. 347.

In 1894–98 Mr J. Romilly Allen was appointed to visit and make outline drawings or photographs of the Sculptured Stones in Scotland for the work on the Early Christian Monuments of Scotland to be issued by the Society, of which about 700 pages have been printed with nearly 2000 illustrations.

In 1899 Mr F. R. Coles was appointed to commence a survey of the Stone Circles in the north-east of Scotland. His Report, with measured plans and drawings of upwards of twenty circles in and near the valley of the Dee, is printed in the *Proceedings*, vol. xxxiv. p. 139.

In 1890 Mr F. R. Coles was again appointed to continue the survey of the Stone Circles of Scotland. His Report, including measured plans and drawings of over twenty circles in and near

the valley of the Don, will be issued in the *Proceedings*, vol. xxxv.

The following extract minute is from the Records of New College Senatus, March 19, 1890:—"The Secretary submitted to the Senatus a bond for One thousand pounds (£1000) by His Excellency Robert Halliday Gunning, Esq., M.D., LL.D., Grand Dignitary of the Empire of Brazil, of Rio de Janeiro and of Edinburgh, in favour of the General Trustees of the Free Church of Scotland, for behoof of the Natural Science Chair, New College, with relative letter from Messrs Auld & Macdonald W.S., Dr Gunning's agents. The objects for which His Excellency has granted this bond are stated in the bond as follows:—'with the view of commemorating the Jubilee of Her Majesty Queen Victoria, and of encouraging the study of Natural Science by students of the Presbyterian Ministry with the view of the defence of the faith when attacked from the scientific standing point; being also desirous of commemorating the name and work of Hugh Miller, and being likewise moved by regard for the present occupant of the Chair (Professor Duns, D.D.) of Natural Science in New College, Edinburgh, I undertake to pay to the General Trustees of the Free Church of Scotland the sum of One thousand pounds (£1000), the income of which is to be placed at the disposal of the Professor of Natural Science in the New College for the time being, to be applied in class prizes, or in purchasing additional objects for the Museum, or scientific appliances or books for the Natural Science Library of the said New College, or in procuring an assistant for the professor.'

"In accepting the very appropriate and handsome gift the Senatus agree to carry out His Excellency's intentions, and they cordially thank him for his thoughtful liberality. They would assure His Excellency that his liberality with the College is highly appreciated both by the Senatus and the Church."

In conclusion, we cannot help acknowledging the value of Dr Gunning's liberality, when under it we have such contributions to the literature of Physics and Archæology.

Dr Gunning died at 12 Addison Crescent, London, on the 22d March 1900. A man valiant for what he held to be true. Acquaintances who knew him best admired him most.

Professor Tait. By Lord Kelvin.

(Read December 2, 1901.)

When Professor Tait last February resigned the chair of Natural Philosophy in the University of Edinburgh, we hoped that the immediate relief from strain and anxiety regarding his duty might conduce to a speedy recovery from the severe illness under which he was then suffering. I was indeed myself sanguine in looking forward to an unbroken continuation of the friendly intercourse with him which I had enjoyed through forty-one years of my life. A slight abatement of the graver symptoms, and a cheering return to some mathematical work left off six months before, gave hope that a change from George Square to Challenger Lodge in June, on the invitation of his friend and former pupil Sir John Murray, might be the beginning of a recovery. But it was not to be. Death came suddenly on the 4th of July, and our friend is gone from us.

Peter Guthrie Tait was born at Dalkeith on 28th April 1831. After early education at Dalkeith Grammar School, and Circus Place School, Edinburgh, he entered the celebrated Edinburgh Academy, of which he remained a pupil till 1847, when he entered the University of Edinburgh. After a session there under Kelland and Forbes, he entered Cambridge in 1848 as an undergraduate of Peterhouse, and in 1852 he took his degree as Senior Wrangler and First Smith's Prizeman, and was elected to a Fellowship of his College. He remained officially in Peterhouse as mathematical lecturer till 1854, when he was called to Queen's College, Belfast, as Professor of Mathematics. This was a most happy appointment for Tait. It made him a colleague of, and co-worker on the electrolytic condensation of mixed oxygen and hydrogen and on ozone with Andrews, the discoverer of a procedure producing continuous change in a homogeneous substance, from liquid to gaseous and from gaseous to liquid condition. Through Andrews it introduced him to William Rowan Hamilton, the discoverer of

the principle of varying action in dynamics, and the inventor of the captivatingly ingenious and beautiful method of quaternions in Mathematics. It gave him six years of good duty in Queen's College, well done, in teaching Mathematics; and for some time also Natural Philosophy, in aid of his colleague Stevelly. During those bright years in Belfast he found his wife, and laid the foundation of a happiness which lasted as long as his life.

In 1860 he was elected to succeed Forbes as Professor of Natural Philosophy in the University of Edinburgh. It was then that I became acquainted with him, and we quickly resolved to join in writing a book on Natural Philosophy, beginning with a purely geometrical preliminary chapter on Kinematics, and going on thence instantly to dynamics, the science of Force, as foundation of all that was to follow. I found him full of reverence for Andrews and Hamilton, and enthusiasm for science. Nothing else worth living for, he said; with heart-felt sincerity I believe, though his life belied the saying, as no one ever was more thorough in public duty or more devoted to family and friends. His two years as "don" of Peterhouse and six of professorial gravity in Belfast had not wholly polished down the rough gaiety nor dulled in the slightest degree the cheerful humour of his student days; and this was a large factor in the success of our alliance for heavy work, in which we persevered for eighteen years. "A merry heart goes all the day, Your sad, tires in a mile-a." The making of the first part of "T and T′" was treated as a perpetual joke, in respect to the irksome details of interchange of drafts for "copy," amendments in type, and final corrections of proofs. It was lightened by interchange of visits between Greenhill Gardens, or Drummond Place, or George Square, and Largs, or Arran, or the old or new College of Glasgow; but of necessity it was largely carried on by post. Even the postman laughed when he delivered one of our missives, about the size of a postage stamp, out of a pocket handkerchief in which he had tied it, to make sure of not dropping it on the way.

One of Tait's humours was writing in charcoal on the bare plaster wall of his study in Greenhill Gardens a great table of living scientific worthies *in order of merit.* Hamilton, Faraday, Andrews, Stokes, and Joule headed the column, if I remember

right. Clerk Maxwell, then a rising star of the first magnitude in our eyes, was too young to appear on the list.

About 1878 we got to the end of our "Division II." on "Abstract Dynamics"; and, according to our initial programme, should then have gone on to "properties of matter," "heat," "light," "electricity," "magnetism." Instead of this we agreed that for the future we could each work more conveniently and on more varied subjects, without the constraint of joint effort to produce as much as we could of an all-comprehensive text-book of Natural Philosophy. Thus our book came to an end with only a foundation laid for our originally intended structure.

Tait's first published work was undertaken in conjunction with a Peterhouse friend, Steele, who was his second in the University both as Wrangler and Smith's Prizeman. They commenced their work together immediately after taking their degrees; but Steele died before more than two or three chapters had been written, and Tait finished it alone, and published it four years later under the title "Tait and Steele's Dynamics of a Particle" (1856). It has gone through many editions, and still holds its place as a text-book.

Tait's second published book, "Elements of Quaternions," was commenced under the auspices of Hamilton; but, in deference to his wish, not published till 1867. It has gone through three editions, and is, I believe, the text-book for all those who wish to learn the subject.

Tait also produced several valuable *Treatises*, short, *readable*, interesting, and useful, on various subjects in physical science:—

"Sketch of Thermodynamics" (1867).

"Recent Advances in Physical Science" (1876).

"Heat" (1884, 2nd edition 1892).

"Light" (1884, 3rd edition 1900), based on article in *Encyclopædia Britannica.*

"Properties of Matter" (1885, 4th edition 1899).

"Dynamics" (1895), based on article "Mechanics" in *Ency. Brit.*

Among smaller articles contributed to the *Ency. Brit.* are "Quaternions," "Radiation and Convection," and "Thermodynamics," all reprinted in the collected papers. A small 50-page book on "Newton's Laws of Motion" is a remarkably concise

statement of the foundations of dynamical science. It is Tait's last published work, primarily intended as a help to medical students attending his special three months' course of lectures for them on Natural Philosophy.

In the Royal Society of Edinburgh we all know something of how Tait has enriched its Proceedings and Transactions by his interesting and varied papers on mathematical and physical subjects from year to year since 1860, when he came to Edinburgh to succeed Forbes as Professor of Natural Philosophy in the University. Nearly all of these are now collected, along with a considerable number of other scientific papers which he brought out through other channels, arranged in order of time, from 1859 to 1898; one hundred and thirty-three articles in all; republished by the Cambridge University Press in two splendid quarto volumes of 500 pages each; a worthy memorial of a life of laborious whole-hearted devotion to science.

The "Scientific Papers" collected in these two volumes abound in matter of permanent scientific interest; and literary interest too, as witness the short articles on "Hamilton," "Macquorne Rankine," "Balfour Stewart," "Clerk Maxwell," and "The Teaching of Natural Philosophy." Of all the mathematical papers in the collection, one of those which seem to me most fundamentally important is Part IV. of "Foundations of the Kinetic Theory of Gases," in which we find the first proof (and, I believe, the only proof hitherto given) of the theorem enunciated first by Waterston and twelve years later independently by Clerk Maxwell, asserting equal average partition of energy between two sets of masses larger and smaller, taken as hard globes to represent the molecules of two different gases thoroughly mixed together. The collection contains also papers describing valuable experimental researches made by Tait through many years on various subjects: Thermo-electricity; Thermal Conductivity of Metals; Impact and Duration of Impact; Pressure Errors of the Challenger thermometers; Compressibility of Water, Glass, and Mercury (contributed originally to the "Physics and Chemistry" of H.M.S. Challenger). His work for the Challenger Report was a splendid series of very difficult experimental researches carried on for about nine years (1879 to 1888), with admirable scientific inventiveness, and no less admirable

zeal and perseverance. One little scientific bye-product of extreme interest I cannot refrain from quoting. Referring to a hermetically sealed glass tube under tests for strength to resist great water pressure, "I enclosed the glass tube in a tube of stout brass, " closed at the bottom only, but was surprised to find that it was " crushed almost flat on the first trial [when the glass tube broke]. " This was evidently due to the fact that water is compressible, " and therefore the relaxation of pressure (produced by the break- " ing of the glass tube) takes time to travel from the inside to the " outside of the brass tube; so that for about 1/10000th of a " second that tube was exposed to a pressure of four or five tons " weight per square inch on its outer surface, and no pressure on " the inner. The impulsive pressure on the bottom of the tube " projected it upwards so that it stuck in the tallow which fills " the hollow of the steel plug. Even a piece of gun-barrel, which " I substituted for the brass tube, was cracked, and an iron disc, " tightly screwed into the bottom of it to close it, was blown in. " I have since used a portion of a thicker gun-barrel, and have had " the end welded in. But I feel sure that an impulsive pressure " of ten or twelve tons weight would seriously damage even this. " These remarks seem to be of interest on several grounds, for they " not only explain the crushing of the open copper cases of those " of the Challenger thermometers which gave way at the bottom " of the sea, but they also give a hint explanatory of the very " remarkable effects of dynamite and other explosives when fired " in the open air. (It is easy to see that, *ceteris paribus*, the " effects of this impulsive pressure will be greater in a large " apparatus than in a small one)."

In a communication on "Charcoal Vacua" to the Royal Society of Edinburgh of July 5, 1875, imperfectly reported in *Nature* of July 15 of that year, the true dynamical explanation of one of the most interesting and suggestive of all the scientific wonders of the nineteenth century, Crookes' radiometer, was clearly given. The phenomenon to be explained is that in highly rarefied air a disc of pith or cork or other substance of small thermal conductivity, blackened on one side, and illuminated by light on all sides, even the cool light of a wholly clouded sky, experiences a steady measurable pressure on the blackened side. Many naturalists, I

believe, had truly attributed this fact to the blackened side being rendered somewhat warmer by the light; but none before Tait and Dewar had ever imagined the dynamical cause,—the largeness of the free path of the molecule of the highly rarefied air, and the greater average velocity of rebound of the molecules from the warmer side. *Long free path* was the open sesame to the mystery.

The Keith Medal of the Royal Society of Edinburgh was awarded to Professor Tait in the year 1869, and again in 1874; and one of the Royal Medals of the Royal Society of London was awarded to him in the year 1886. The Gunning Victoria Jubilee Prize of the Royal Society of Edinburgh was awarded to him in 1890.

Enthusiast as he was in experimental and mathematical work, he never allowed this to interfere with his University teaching, to which, from beginning to end of the forty years of his Professorship, he devoted himself with ever fresh vigour, and with unremitting faithfulness, as his primary public duty. How happily and usefully and inspiringly he performed it, has been remembered with gratitude by all who have ever had the privilege of being students in his class.

With not less devotion and faithfulness during all these years he has worked for the Royal Society, of which he was elected a Fellow when he came to Edinburgh as Professor. At the commencement of the following session he was elected a Member of Council; and in 1864 he became one of the Secretaries to the ordinary meetings. In 1879, in succession to Professor Balfour, he was elected to the General Secretaryship; and he held this office till the end of his life.

His loss will be felt in the Society, not only as an active participator in its scientific work, but also as a wise counsellor and guide. It has been put on record that "The Council always felt that in "his hands the affairs of the Society were safe, that nothing would "be forgotten, and that everything that ought to be done would be "brought before it at the right time and in the right way." In words that have already been used by the Council, I desire now to say on the part, not only of the Council, but of all who have known Tait personally, and of a largely wider circle of scientific men who know his works,—"We all feel that a great man has

"been removed; a man great in intellect, and in the power of using "it, and in clearness of vision and purity of purpose, and therefore "great in his influence, always for good, on his fellowmen; we feel "that we have lost a strong and true friend."

After enjoying eighteen years' joint work with Tait on our book, twenty-three years without this tie have given me undiminished pleasure in all my intercourse with him. I cannot say that our meetings were never unruffled. We had keen differences (much more frequent agreements) on every conceivable subject,—quaternions, energy, the daily news, politics, *quicquid agunt homines*, etc., etc. We never agreed to differ, always fought it out. But it was almost as great a pleasure to fight with Tait as to agree with him. His death is a loss to me which cannot, as long as I live, be replaced.

The cheerful brightness which I found on our first acquaintance forty-one years ago remained fresh during all these years, till first clouded when news came of the death in battle of his son Freddie in South Africa, on the day of his return to duty after recovery from wounds received at Magersfontein. The cheerfulness never quite returned. The sad and final break-down in health came after a few weeks of his University lectures in October and November of last year. His last lecture was given on December 11, 1900.

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